Maths-2 Class on Zoom Date-28.03.2020

Divergence of a Vector Field

If $\vec{V}(x,y,z) = V_1(x,y,z)\hat{i} + V_2(x,y,z)\hat{j} + V_3(x,y,z)\hat{k}$ is a vector function and $\vec{\nabla} = \frac{\partial}{\partial x}\hat{i} + \frac{\partial}{\partial y}\hat{j} + \frac{\partial}{\partial z}\hat{k}$ is the vector differential operator, then the divergence of $\vec{V}$ which is denoted as div $\vec{V}$ and is defined as

$$\text{div.}\vec{V} = \vec{\nabla}\cdot\vec{V} = \left(\frac{\partial}{\partial x}\hat{i} + \frac{\partial}{\partial y}\hat{j} + \frac{\partial}{\partial z}\hat{k}\right)\cdot(V_1\hat{i} + V_2\hat{j} + V_3\hat{k})$$

ie $$\boxed{\text{div}\,\vec{V} = \frac{\partial V_1}{\partial x} + \frac{\partial V_2}{\partial y} + \frac{\partial V_3}{\partial z}}$$ ($\because$ dot product $\vec{a}\cdot\vec{b} = a_1a_2 + b_1b_2 + c_1c_2$)

Note: As divergence of $\vec{V}$ is a dot product of $\vec{\nabla}$ & $\vec{V}$, it is no doubt a scalar function/output.

Examples

Ex-1 Find div $\vec{V}$, where $\vec{V} = 3xz\hat{i} + 2xy\hat{j} - yz^2\hat{k}$

Sol^n $\text{div}\,\vec{V} = \frac{\partial V_1}{\partial x} + \frac{\partial V_2}{\partial y} + \frac{\partial V_3}{\partial z}$, Here $V_1 = 3xz$, $V_2 = 2xy$, $V_3 = -yz^2$

Now $\frac{\partial V_1}{\partial x} = \frac{\partial}{\partial x}(3xz) = 3z$

$\frac{\partial V_2}{\partial y} = \frac{\partial}{\partial y}(2xy) = 2x$

$\frac{\partial V_3}{\partial z} = \frac{\partial}{\partial z}(-yz^2) = -2yz$

$\therefore$ $\boxed{\text{div}\,\vec{V} = 3z + 2x - 2yz}$ Ans (scalar function)

Q Find divergence of the following vector functions.

(a) $x^2\hat{i} + y^2\hat{j} + z^2\hat{k}$

(b) $e^x(\cos y\,\hat{i} + \sin y\,\hat{j})$

(c) $xyz(x\hat{i} + y\hat{j} + z\hat{k})$

Sol^n (a) Here $\vec{V} = x^2\hat{i} + y^2\hat{j} + z^2\hat{k}$, i.e. $V_1 = x^2$, $V_2 = y^2$, $V_3 = z^2$

Now $\frac{\partial V_1}{\partial x} = 2x$, $\frac{\partial V_2}{\partial y} = 2y$, $\frac{\partial V_3}{\partial z} = 2z$

$\therefore$ $\text{div}\,\vec{V} = \frac{\partial V_1}{\partial x} + \frac{\partial V_2}{\partial y} + \frac{\partial V_3}{\partial z} = 2x + 2y + 2z = 2(x+y+z)$ Ans.

(b) $\vec{V} = e^{x}(\cos y\,\hat{i} + \sin y\,\hat{j})$ i.e. $v_1 = e^{x}\cos y$, $v_2 = e^{x}\sin y$, $v_3 = 0$

$$\frac{\partial v_1}{\partial x} = \frac{\partial}{\partial x}(e^{x}\cos y) = e^{x}\cos y$$

$$\frac{\partial v_2}{\partial y} = \frac{\partial}{\partial y}(e^{x}\sin y) = e^{x}\cos y, \quad \frac{\partial v_3}{\partial z} = 0$$

$$\therefore \operatorname{div}\vec{V} = \frac{\partial v_1}{\partial x} + \frac{\partial v_2}{\partial y} + \frac{\partial v_3}{\partial z} = e^{x}\cos y + e^{x}\cos y + 0 = \underline{2e^{x}\cos y} \text{ Ans}$$

(c) $\vec{V} = xyz(x\hat{i} + y\hat{j} + z\hat{k})$, i.e. $v_1 = x^2yz$, $v_2 = xy^2z$, $v_3 = xyz^2$

Now $\frac{\partial v_1}{\partial x} = \frac{\partial}{\partial x}(x^2yz) = 2xyz$

$$\frac{\partial v_2}{\partial y} = \frac{\partial}{\partial y}(xy^2z) = 2xyz$$

$$\frac{\partial v_3}{\partial y} = \frac{\partial}{\partial y}(xyz^2) = 2xyz$$

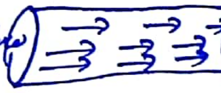
$$\therefore \operatorname{div}\vec{V} = \frac{\partial v_1}{\partial x} + \frac{\partial v_2}{\partial y} + \frac{\partial v_3}{\partial z} = 2xyz + 2xyz + 2xyz = \underline{6xyz} \text{ Ans.}$$

Application (Physical) in Fluid Dynamics

The divergence of the velocity function in the motion of fluids are often used to express the governing equation i.e. equation of continuity. The eqn of continuity represents the conservation of mass i.e. Inflow = out flow.

It is represented as:

$$\boxed{\frac{\partial \rho}{\partial t} + \operatorname{div}(\rho V) = 0} \quad \text{— (A)}$$

Inflow  out flow

Here ρ = density (mass per unit volume)

If the flow is steady i.e. time independent $\Rightarrow \frac{\partial \rho}{\partial t} = 0$

Thus eqn of continuity reduces to

$$\operatorname{div}(\rho\vec{V}) = 0 \quad \text{— (B)}$$

If the fluid is incompressible, the ρ = const.,

Thus eqn-B reduces to

$$\rho\,\operatorname{div}(\vec{V}) = 0 \quad \text{— (C)}$$

$$\Rightarrow \boxed{\operatorname{div}(\vec{V}) = 0} \quad (\because \rho \neq 0)$$

Now eqn-C represents the eqn of continuity in the motion of incompressible fluids.

Note: If $\operatorname{div}\vec{V} = 0$ i.e. divergence of velocity fn. is 0, then the fluid is incompressible otherwise compressible.

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Examples

Q. Determine weather the fluid is incompressible or not if its velocity function is given as:

(a) $V = x\hat{i} + y\hat{j} - z\hat{k}$ (b) $V = -\frac{y}{4}\hat{i} + 4x\hat{j}$ (c) $V = 2y^2\hat{i}$, (d) $y\hat{i} - x\hat{j}$.

Sol^n (a) $V = x\hat{i} + y\hat{j} - z\hat{k}$

Now $\operatorname{div}\vec{V} = \frac{\partial V_1}{\partial x} + \frac{\partial V_2}{\partial y} + \frac{\partial V_3}{\partial z}$

$= \frac{\partial}{\partial x}(x) + \frac{\partial}{\partial y}(y) - \frac{\partial}{\partial z}(z)$

$= 1 + 1 - 1 = 1 \neq 0$

$\therefore$ The fluid is compressible.

(b) $V = -\frac{y}{4}\hat{i} + 4x\hat{j}$

$\operatorname{div}\vec{V} = \frac{\partial V_1}{\partial x} + \frac{\partial V_2}{\partial y} + \frac{\partial V_3}{\partial z} = \frac{\partial}{\partial x}\left(-\frac{y}{4}\right) + \frac{\partial}{\partial y}(4x) + \frac{\partial}{\partial z}(0)$

$= 0 + 0 + 0 = 0$

$\therefore$ Fluid is incompressible.

(c) $\vec{V} = 2y^2\hat{i}$

$\operatorname{div}\vec{V} = \frac{\partial V_1}{\partial x} + \frac{\partial V_2}{\partial y} + \frac{\partial V_3}{\partial z} = \frac{\partial}{\partial x}(2y^2) + 0 + 0 = 0$

$\therefore$ Fluid is incompressible.

(d) $\vec{V} = y\hat{i} - x\hat{j}$

$\therefore \operatorname{div}\vec{V} = \frac{\partial V_1}{\partial x} + \frac{\partial V_2}{\partial y} + \frac{\partial V_3}{\partial z} = \frac{\partial}{\partial x}(y) - \frac{\partial}{\partial y}(x) = 0 - 0 = 0$

$\therefore$ Fluid is incompressible.

Note As gradient of a scalar function $f = f(x,y,z)$ is a vector function

$\Rightarrow$ The divergence of (grad f) exists and it is a scalar fn.

Now $\operatorname{div}(\operatorname{grad} f) = \vec{\nabla}\cdot(\vec{\nabla}f) = \left(\frac{\partial}{\partial x}\hat{i} + \frac{\partial}{\partial y}\hat{j} + \frac{\partial}{\partial z}\hat{k}\right)\cdot\left(\frac{\partial f_1}{\partial x}\hat{i} + \frac{\partial f_2}{\partial y}\hat{j} + \frac{\partial f_3}{\partial z}\right)$

$\therefore \operatorname{div}(\operatorname{grad} f) = \frac{\partial^2 f_1}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2} = \nabla^2 f$ (Laplacian operator)

where $\boxed{\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}}$ The second order partial differential operator.

Some useful Formulas

1. $\operatorname{div}(kV) = k \operatorname{div}\vec{V}$ 2. $\operatorname{div}(fV) = f\operatorname{div}\vec{V} + \nabla\cdot\nabla f$.
3. $\operatorname{div}(f\nabla g) = f\nabla^2 g + \nabla f\cdot\nabla g$

(Prove it) Assignment

Q Find the Laplacian of the following functions.
(a) $f = 4x^2 + 9y^2 + z^2$ (b) $f = e^{2x}\sin 2y$ (c) $f = \frac{xy}{z}$.

Sol^n (a) Laplacian of $f = \nabla^2 f = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2}$

$f = 4x^2 + 9y^2 + z^2$

$\therefore \frac{\partial^2 f}{\partial x^2} = 8, \quad \frac{\partial^2 f}{\partial y^2} = 18, \quad \frac{\partial^2 f}{\partial z^2} = 2$

$\therefore \nabla^2 f = 8 + 18 + 2 = 28$

(b) $f = e^{2x}\sin 2y$

$\frac{\partial f}{\partial x} = 2e^{2x}\sin 2y \qquad \frac{\partial f}{\partial y} = 2e^{2x}\cos 2y \qquad \frac{\partial^2 f}{\partial z^2} = 0$

$\frac{\partial^2 f}{\partial x^2} = 4e^{2x}\sin 2y \qquad \frac{\partial^2 f}{\partial y^2} = -4e^{2x}\sin 2y.$

$\therefore \nabla^2 f = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z} = 4e^{2x}\sin 2y - 4e^{2x}\sin 2y + 0 = 0$

(c) $f = \frac{xy}{z}$

$\frac{\partial f}{\partial x} = \frac{y}{z} \qquad \frac{\partial f}{\partial y} = \frac{x}{z} \qquad \frac{\partial f}{\partial z} = -\frac{xy}{z^2}$

$\frac{\partial^2 f}{\partial x^2} = 0 \qquad \frac{\partial^2 f}{\partial z^2} = 0 \qquad \frac{\partial^2 f}{\partial z^2} = \frac{2xy}{z^3}$

$\therefore \nabla^2 f = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2} = 0 + 0 + \frac{2xy}{z^3} = \frac{2xy}{z^3}$

Assignments

Q Find the divergence of the following vector functions.
(a) $\vec{V} = x\hat{i} + y\hat{j} + z\hat{k}$ (b) $\frac{-y}{x^2+y^2}\hat{i} + \frac{x}{x^2+y^2}\hat{j}$
(c) $\vec{V} = \frac{x}{\sqrt{x^2+y^2+z^2}}\hat{i} + \frac{y}{\sqrt{x^2+y^2+z^2}} + \frac{z}{\sqrt{x^2+y^2+z^2}}\hat{k}$ (d) $\vec{V} = e^{x}\hat{i} + ye^{-x}\hat{j} + 2z\sinh x\,\hat{k}$

Q Find the Laplacian ($\nabla^2 f$) for the following fns.
(a) $f = \frac{x-y}{x+y}$ (b) $f = \cos^2 x - \sin^2 y$ (c) $f = z - \sqrt{x^2+y^2}$ (d) $f = \tan^{-1}\left(\frac{y}{x}\right)$

Q- Determine weather the fluids are incompressible or not whose velocity functions are.
(a) $V = x^3\hat{k}$ (b) $\vec{V} = \sec x\,\hat{i} + \operatorname{cosec} x\,\hat{j}$ (c) $\sin y\,\hat{i} + \cos z\,\hat{k}$

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Curl of Vector Functions

The curl of a vector function $\vec{V} = V_1(x,y,z)\hat{i} + V_2(x,y,z)\hat{j} + V_3(x,y,z)\hat{k}$ is a vector fn and is defined & denoted as

$$\text{Curl}\,\vec{V} = \vec{\nabla}\times\vec{V} = \left(\frac{\partial}{\partial x}\hat{i} + \frac{\partial}{\partial y}\hat{j} + \frac{\partial}{\partial z}\hat{k}\right)\times\left(V_1\hat{i} + V_2\hat{j} + V_3\hat{k}\right)$$

$$\therefore \boxed{\text{Curl}\,\vec{V} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ V_1 & V_2 & V_3 \end{vmatrix} = \left(\frac{\partial V_3}{\partial y} - \frac{\partial V_2}{\partial z}\right)\hat{i} + \left(\frac{\partial V_1}{\partial z} - \frac{\partial V_3}{\partial x}\right)\hat{j} + \left(\frac{\partial V_2}{\partial x} - \frac{\partial V_1}{\partial y}\right)\hat{k}}$$

Examples: Find curl for the following vector functions.

(a) $V = yz\hat{i} + 3zx\hat{j} + z\hat{k}$ (b) $2y\hat{i} + 5x\hat{j}$

(c) $xyz(x\hat{i} + y\hat{j} + z\hat{k})$ (d) $\frac{1}{2}(x^2+y^2+z^2)(\hat{i}+\hat{j}+\hat{k})$

Sol^n(a) $\vec{V} = yz\hat{i} + 3zx\hat{j} + z\hat{k}$

$\Rightarrow V_1 = yz,\ V_2 = 3zx,\ V_3 = z$

$$\therefore \text{Curl}\,\vec{V} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ yz & 3zx & z \end{vmatrix} = \left(\frac{\partial}{\partial y}(z) - \frac{\partial}{\partial z}(3zx)\right)\hat{i} + \left\{\frac{\partial}{\partial z}(yz) - \frac{\partial}{\partial x}(z)\right\}\hat{j} + \left\{\frac{\partial}{\partial x}(3zx) - \frac{\partial}{\partial y}(yz)\right\}\hat{k}$$

$$= (0-3x)\hat{i} + (y-0)\hat{j} + (3z - z)\hat{k}$$

$$\therefore \boxed{\text{Curl}\,\vec{V} = -3x\hat{i} + y\hat{j} + 2z\hat{k}} \text{ Ans.}$$

(b) $V = 2y\hat{i} + 5x\hat{j}$

$$\text{Curl}\,\vec{V} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 2y & 5x & 0 \end{vmatrix} = \left\{\frac{\partial}{\partial y}(0) - \frac{\partial}{\partial z}(5x)\right\}\hat{i} + \left\{\frac{\partial}{\partial z}(2y) - \frac{\partial}{\partial x}(0)\right\}\hat{j} + \left\{\frac{\partial}{\partial x}(5x) - \frac{\partial}{\partial y}(2y)\right\}\hat{k}$$

$$\therefore \text{Curl}\,\vec{V} = 0\hat{i} + 0\hat{j} + (5-2)\hat{k} = [0, 0, 3]$$

(c) $\vec{V} = xyz(x\hat{i} + y\hat{j} + z\hat{k}) = x^2yz\hat{i} + xy^2z\hat{j} + xyz^2\hat{k}$

$$\text{Curl}\,\vec{V} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x^2yz & xy^2z & xyz^2 \end{vmatrix} = \left\{\frac{\partial}{\partial y}(xyz^2) - \frac{\partial}{\partial z}(xy^2z)\right\}\hat{i} + \left\{\frac{\partial}{\partial z}(x^2yz) - \frac{\partial}{\partial x}(xyz^2)\right\}\hat{j} + \left\{\frac{\partial}{\partial x}(xy^2z) - \frac{\partial}{\partial y}(x^2yz)\right\}\hat{k}$$

$$= (xz^2 - xy^2)\hat{i} + (x^2y - yz^2)\hat{j} + (y^2z - x^2z)\hat{k}. \text{ Ans.}$$

Application (Physical)

In case of a rotation of a rigid body, if the curl of the velocity field $=0$, the motion of the body is irrotational or else rotational.

Ex Determine weather the motion of a rigid body is rotational or irrotational whose velocity functions are.

(a) $\vec{V} = 2y^2\vec{i}$ (b) $\vec{V} = x^3\vec{k}$ (c) $V = y\vec{i} - x\vec{j}$.

Sol^n

(a) $\vec{V} = 2y^2$

$$\text{curl}\, v = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 2y^2 & 0 & 0 \end{vmatrix} = \left(\frac{\partial}{\partial y}(0) - \frac{\partial}{\partial z}(0)\right)\vec{i} + \left(\frac{\partial}{\partial z}(2y^2) - \frac{\partial}{\partial x}(0)\right)\vec{j} + \left(\frac{\partial}{\partial x}(0) - \frac{\partial}{\partial y}(2y^2)\right)$$

$$= 0\vec{i} + 0\vec{j} - 4y\vec{k} \neq 0 \Rightarrow \text{rotational.}$$

(b) $\vec{V} = x^3\vec{k}$,

$$\text{curl}\,\vec{V} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 0 & 0 & x^3 \end{vmatrix} = \left(\frac{\partial}{\partial y}(x^3) - \frac{\partial}{\partial z}(0)\right)\vec{i} + \left(\frac{\partial}{\partial z}(0) - \frac{\partial}{\partial x}(x^3)\right)\vec{j}$$
$$+ \left(\frac{\partial}{\partial x}(0) - \frac{\partial}{\partial y}(0)\right)\vec{k} = -3x^2\vec{j}$$

$\Rightarrow$ rotational.

Note-1 $\boxed{\text{curl}(\text{grad} f) = 0}$

Pf
$$\text{curl}(\text{grad} f) = \vec{\nabla} \times \left(\frac{\partial f}{\partial x}\vec{i} + \frac{\partial f}{\partial y}\vec{j} + \frac{\partial f}{\partial z}\vec{k}\right) = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} & \frac{\partial f}{\partial z} \end{vmatrix}$$
$$= \left(\frac{\partial^2 f}{\partial y \partial z} - \frac{\partial^2 f}{\partial z \partial y}\right)\vec{i} + \left(\frac{\partial^2 f}{\partial z \partial x} - \frac{\partial^2 f}{\partial x \partial z}\right) + \left(\frac{\partial^2 f}{\partial x \partial y} - \frac{\partial^2 f}{\partial y \partial x}\right)\vec{k} = 0.$$

Note-2 $\boxed{\text{div}(\text{curl}\,\vec{V}) = 0}$ Similar proof

Assignments

Q Find curl of the following functions.

(a) $\vec{V} = \frac{x}{\sqrt{x^2+y^2+z^2}}\vec{i} + \frac{y}{\sqrt{x^2+y^2+z^2}}\vec{j} + \frac{z}{\sqrt{x^2+y^2+z^2}}\vec{k}$

(b) $xyz(x^2\vec{i} + y^2\vec{j} + z^2\vec{k})$ (c) $\frac{1}{2}(x^2+y^2+z^2)(\vec{i} + \vec{j} + \vec{k})$

Q Determine weather the motion of the rigid body is rotational or not. whose velocity functions are.

(a) $\vec{V} = \sec x\,\vec{i} + \operatorname{cosec} x\,\vec{j}$ (b) $x\vec{i} + y\vec{j} + z\vec{k}$ (c) $y\vec{i} - z\vec{j} + x\vec{k}$

Q Prove the followings.

(a) $\text{Curl}(\vec{u} + \vec{v}) = \text{curl}\,\vec{u} + \text{curl}\,\vec{v}$

(b) $\text{curl}(\text{grad} f) = 0$ (c) $\text{div}(\text{curl}\,\vec{v}) = 0$

(d) $\text{div}(\vec{u} \times \vec{v}) = \vec{v} \cdot \text{curl}\,\vec{u} - u \cdot \text{curl}\,\vec{v}$

(Hints: Take $u = u_1\vec{i} + u_2\vec{j} + u_3\vec{k}$, $V = v_1\vec{i} + v_2\vec{j} + v_3\vec{k}$)
& then prove LHS = RHS

Vector Calculus: Line Integrals

The concept of line integrals is a generalization of definite integrals i.e. $\int_a^b f(x)dx$. (integration betn $[a,b]$)

The line integrals are represented by

$$\boxed{\int_C F(r)\cdot dr \quad \text{or} \quad \oint_C F(r)\cdot dr}$$

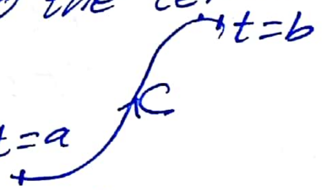
Here the vector function $F(r) = F_1(x,y,z)\vec{i} + F_2(x,y,z)\vec{j} + F_3(x,y,z)\vec{k}$ is integrated along a curve called path 'C' instead of an interval (a,b).

In the second case the path of the integral is a closed curve.

The curve or path of the integral usually represented in its parametric form as:

$$\boxed{C: r(t) = x(t)\vec{i} + y(t)\vec{j} + z(t)\vec{k}}, \ t \text{ is a parameter. } a \le t \le b$$

Here a is the value of 't' corresponding to the initial point & b is the value of 't' corresponding to the terminal or end point on 'C'.



Parametric Representation of some Known curves

Name of the curve	Parametric Representation
1. Straight line	$r(t) = \vec{a} + t\vec{b}$ (Passing throug $\vec{a}$ & $\parallel^{el}$ to $\vec{b}$)
2. Circle $(x^2+y^2=a^2)$ →	$r(t) = a\cos t\,\vec{i} + a\sin t\,\vec{j}$ $(0 \le t \le 2\pi)$ (center at $(0,0)$ & radius $= a$)
3. Ellipse $\left(\frac{x^2}{a^2}+\frac{y^2}{b^2}=1\right)$ →	$r(t) = a\cos t\,\vec{i} + b\sin t\,\vec{j}$ (")
4. Parabola $(y^2 = 4ax)$ →	$r(t) = at^2\vec{i} + 2at\,\vec{j}$
5. Circular helix →	$r(t) = a\cos t\,\vec{i} + a\sin t\,\vec{j} + ct\,\vec{k}$

etc.

N.B Here all the curves discussed here are assumed to be piecewise smooth.

Representation of Line Integrals

$$\int_C F(r)\cdot dr = \int_C (F_1\,dx + F_2\,dy + F_3\,dz)$$
$$= \int_a^b (F_1 x' + F_2 y' + F_3 z')\,dt.$$

Evaluation of Line Integrals

Working Formula: $\boxed{\int_C F(\vec{r})\cdot d\vec{r} = \int_a^b F(\vec{r}(t))\cdot\left(\frac{d\vec{r}}{dt}\right)dt.}$ —(1)

Working Procedure:

Step-1: Write or note down the parametric representation of the curve 'C'
i.e. $C: r(t) = x(t)\hat{i} + y(t)\hat{j} + z(t)\hat{k}$ & note down $x(t), y(t), z(t)$.

Step-2: Find $F(r(t))$ i.e. replace x, y & z of 'F' by $x(t), y(t)$ & $z(t)$ respectively from 'C': $r(t)$ (obtained as in step-1)

Step-3: Differentiate $r(t)$ w.r.t. 't' to get $\frac{dr}{dt}$
i.e. $\frac{dr}{dt} = \frac{dx(t)}{dt}\hat{i} + \frac{dy(t)}{dt}\hat{j} + \frac{d}{dt}z(t)\hat{k}$

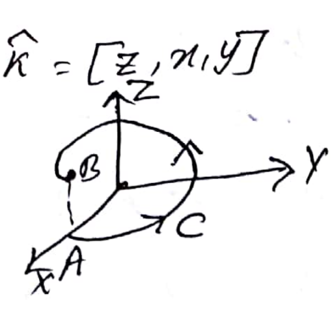
Step-4: Note down the initial and terminal points on C
i.e. $t = a$ & $t = b$ respectively.

Step-5: Find the dot product of $F(r(t))\cdot\frac{dr}{dt}$
($F(r(t))$ from step-2 & $\frac{dr}{dt}$ from step-3)

Step-6 Integrate & evaluate as
$\int_C F\cdot dr = \int_a^b F(r(t))\cdot\frac{dr}{dt}\,dt$ over the limits $t=a$ to $t=b$.

Examples

Q1 Evaluate $\int_C F\cdot d\vec{r}$, where $F(r) = z\hat{i} + x\hat{j} + y\hat{k} = [z, x, y]$
and $C: r(t) = \cos t\,\hat{i} + \sin t\,\hat{j} + 3t\,\hat{k}$ $(0 \le t \le 2\pi)$



Sol^n Parametric representation of 'C' is
$r(t) = \cos t\,\hat{i} + \sin t\,\hat{j} + 3t\,\hat{k}$ (step-1)
$\Rightarrow x = \cos t,\ y = \sin t,\ z = 3t$ —(1)
Now $F = z\hat{i} + x\hat{j} + y\hat{k}$ —(2)
$\Rightarrow F(r(t)) = 3t\,\hat{i} + \cos t\,\hat{j} + \sin t\,\hat{k}$ (from eq^n 1)
(Replacing x, y, z of F by x, y, z of C) (step-2)

Now $r(t) = \cos t\,\hat{i} + \sin t\,\hat{j} + 3t\,\hat{k}$ (step-3)

$\Rightarrow \frac{d\vec{r}}{dt} = -\sin t\,\hat{i} + \cos t\,\hat{j} + 3\hat{k}$. — (3)

The representation 'C' confirms that $\boxed{a=0,\ b=2\pi}$ ($\because 0 \le t \le 2\pi$) (step-4)

Now the dot product of eq 2 & 3.

$F(r(t)) \cdot \frac{dr}{dt} = (3t\,\hat{i} + \cos t\,\hat{j} + \sin t\,\hat{k}) \cdot (-\sin t\,\hat{i} + \cos t\,\hat{j} + 3\hat{k})$ (step-5)

$\therefore \boxed{F(r(t)) \cdot \frac{dr}{dt} = -3t\sin t + \cos^2 t + 3\sin t}$ ($\because a \cdot b = a_1a_2 + b_1b_2 + c_1c_2$)

Thus $\int_C \vec{F} \cdot d\vec{r} = \int_a^b F(r(t)) \cdot \frac{dr}{dt}\,dt$ (step-6)

$$= \int_0^{2\pi} (-3t\sin t + \cos^2 t + 3\sin t)\,dt.$$

$$= (-3)\left\{[-t\cos t]_0^{2\pi} + \int_0^{2\pi} \cos t\,dt\right\} + \frac{1}{2}\left\{\int_0^{2\pi} dt + \left[\frac{\sin 2t}{2}\right]_0^{2\pi}\right\} + 3\,[-\cos t]_0^{2\pi}$$

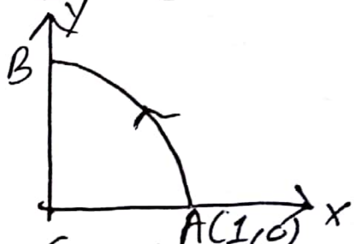
$$= (-3)\{(-2\pi - 0) + 0\} + \frac{1}{2}(2\pi + 0) - 3(1-1)$$

$$= 6\pi + \pi + 0$$

$$= 7\pi$$

$\therefore \boxed{\int_C \vec{F} \cdot d\vec{r} = 7\pi}$ Ans.

Q. Find $\int_C \vec{F} \cdot d\vec{r}$ where $F(r) = [-y, -xy]$ and 'C' is the circular arc as shown in the fig.



(radius $= a = 1$)

Sol^n From the figure the parametric representation of 'C' is

$C: r(t) = \cos t\,\hat{i} + \sin t\,\hat{j}$ — (1) $\left(0 \le t \le \frac{\pi}{2}\right)$ (step-1)

$\Rightarrow x(t) = \cos t$ & $y(t) = \sin t$, $z(t) = 0$.

Now $F(r) = -y\,\hat{i} - xy\,\hat{j}$

$\Rightarrow \boxed{F(r(t)) = -\sin t\,\hat{i} - \sin t \cdot \cos t\,\hat{j}}$ — (2) (step-2)

(PP-4)

$r(t) = \cos t\,\hat{i} + \sin t\,\hat{j}$

Differentiating w.r.t. 't'

(step-3)

$$\boxed{\frac{dr}{dt} = -\sin t\,\hat{i} + \cos t\,\hat{j}} \quad \text{--- (3)}$$

From the figure of curve 'c' it is clear that

$\boxed{a = 0;\ b = \frac{\pi}{2}}$ ($\because$ Quarter of a circle $0 \le t \le \frac{\pi}{2}$) (step-4)

Now making dot product of eq^{n}s 2 & 3.

$$F(r(t)) \cdot \frac{dr}{dt} = (-\sin t\,\hat{i} - \sin t \cdot \cos t\,\hat{j}) \cdot (-\sin t\,\hat{i} + \cos t\,\hat{j}) \text{ (step}$$

$$\Rightarrow \boxed{F(r(t)) \cdot \frac{dr}{dt} = \sin^2 t - \cos^2 t \cdot \sin t} \quad (\because a \cdot b = a_1a_2 + b_1b_2 + c_1c_2)$$

Now

$$\int_C \vec{F} \cdot d\vec{r} = \int_a^b F(r(t)) \cdot \frac{dr}{dt}\, dt \qquad \text{(step-6)}$$

$$= \int_0^{\frac{\pi}{2}} (\sin^2 t - \cos^2 t \cdot \sin t)\, dt$$

$$= \int_0^{\frac{\pi}{2}} \left(\frac{1 - \cos 2t}{2}\right) dt - \int_0^{\frac{\pi}{2}} \cos^2 t \cdot \sin t\, dt$$

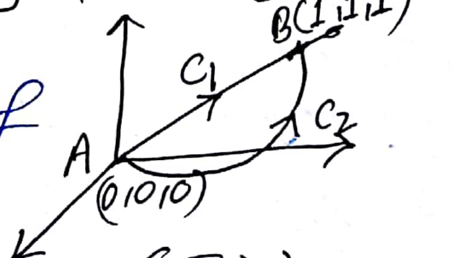
$$= \frac{1}{2}\left\{[t]_0^{\frac{\pi}{2}} - \left[\frac{\sin 2t}{2}\right]_0^{\frac{\pi}{2}}\right\} + \left[\frac{\cos^3 t}{3}\right]_0^{\frac{\pi}{2}}$$

$$= \frac{1}{2}\left\{\left(\frac{\pi}{2} - 0\right) - 0\right\} + \frac{1}{3}(0 - 1) = \frac{\pi}{4} - \frac{1}{3}$$

$$\therefore \boxed{\int_C \vec{F} \cdot d\vec{r} = \frac{\pi}{4} - \frac{1}{3}} \quad \text{Ans} \qquad \boxed{\text{Ans} = \frac{\pi}{4} - \frac{1}{3}}$$

Path Dependence

Evaluate the line integral $\int_C \vec{F} \cdot dr$ along two different Paths C_1 & C_2, where $F = [5z, xy, x^2z]$ & C_1 & C_2 are as shown in the fig.



(Fig)

Sol^n The parametric representations of the curves are.

$C_1: r_1(t) = t\,\hat{i} + t\,\hat{j} + t\,\hat{k} \quad (0 \le t \le 1)$

$C_2: r_2(t) = t\,\hat{i} + t\,\hat{j} + t^2\,\hat{k} \quad (0 \le t \le 1)$

Case-1

Evaluation of $\int_{C_1} \vec{F}\cdot d\vec{r}$ along C_1

Here $r_1(t) = t\hat{i} + t\hat{j} + t\hat{k}$

$\Rightarrow x = t,\ y = t\ \&\ z = t.$

$F = 5z\hat{i} + xy\hat{j} + x^2z\hat{k}$ —(1)

$\Rightarrow \boxed{F(r(t)) = 5t\hat{i} + t^2\hat{j} + t^3\hat{k}}$ (Replacing x, y, z by t, t, t respectively)

Now from the figure $0 \le t \le 1$

Now $r_1(t) = t\hat{i} + t\hat{j} + t\hat{k}$

$\Rightarrow \boxed{\frac{dr_1}{dt} = \hat{i} + \hat{j} + \hat{k}}$ —(2)

Making dot product of 1 & 2

$F(r(t))\cdot\frac{dr}{dt} = (5t\hat{i} + t^2\hat{j} + t^3\hat{k})\cdot(\hat{i} + \hat{j} + \hat{k})$

$\therefore F(r(t))\cdot\frac{dr}{dt} = 5t + t^2 + t^3$

Hence $\int_C \vec{F}\cdot d\vec{r} = \int_C F(r(t))\cdot\frac{dr}{dt}\,dt$

$= \int_0^1 (5t + t^2 + t^3)\,dt$

$= 5\left[\frac{t^2}{2}\right]_0^1 + \left[\frac{t^3}{3}\right]_0^1 + \left[\frac{t^4}{4}\right]_0^1$

$= \frac{5}{2} + \frac{1}{3} + \frac{1}{4} = \frac{30+4+3}{12} = \frac{37}{12}$

Case-2

Evaluation of $\int_{C_2} \vec{F}\cdot d\vec{r}$ along the path C_2.

Parametric representation of C_2 is:

$C_2: r_2(t) = t\hat{i} + t\hat{j} + t^2\hat{k}$

$\Rightarrow x = t,\ y = t,\ z = t^2$

Now $F = 5z\hat{i} + xy\hat{j} + x^2z\hat{k}$

$\therefore \boxed{F(r(t)) = 5t^2\hat{i} + t^2\hat{j} + t^4\hat{k}}$ —(1)

Now from the figure, it is cleare that $0 \le t \le 1$
i.e. $a=0, b=1$

Now $r(t) = t\vec{i} + t\vec{j} + t^2\vec{k}$

$$\therefore \boxed{\frac{dr}{dt} = \vec{i} + \vec{j} + 2t\vec{k}} \text{ --- } (2)$$

Making dot product of 1 & 2

$$F(r(t)) \cdot \frac{dr}{dt} = (5t^2\vec{i} + t^2\vec{j} + t^4\hat{k}) \cdot (\vec{i} + \vec{j} + 2t\hat{k})$$

$$\Rightarrow \boxed{F(r(t)) \cdot \frac{dr}{dt} = 5t^2 + t^2 + 2t^5} \boxed{= 6t^2 + 2t^5}$$

$$\therefore \int_C \vec{F} \cdot d\vec{r} = \int_0^1 F(r(t)) \cdot \frac{dr}{dt}\, dt$$

$$= \int_0^1 (6t^2 + 2t^5)\, dt$$

$$= 6\left[\frac{t^3}{3}\right]_0^1 + \frac{2}{6}\left[t^6\right]_0^1 = 2(1-0) + \frac{1}{3}(1-0) = 2 + \frac{1}{3} = \frac{7}{3} \simeq \frac{28}{12}$$

$$\therefore \boxed{\int_C F \cdot dr = \frac{7}{3} \simeq \frac{28}{12}}$$

N.B. In case-1, the integral value was $\frac{37}{12}$ & now it is $\frac{28}{12}$ which shows that the line Integral may be path dependent.

Assignments

Evaluate $\int_C \vec{F} \cdot d\vec{r}$ along C for the following cases.

1. $F = [y^2, -x^2]$ C: straight line $C\; r(t) = t\vec{i} + 4t\vec{j}$ $(0 \le t \le 1)$
2. $F = [xy, x^2y^2]$ $C: r(t) = 2\cos t\,\vec{i} + 2\sin t\,\vec{j}$ $(0 \le t \le \frac{\pi}{2})$
3. $F = [(x-y)^2, (y-x)^2]$ $C: r(t) = t\vec{i} + \frac{1}{t}\vec{j}$ $0 \le t \le 4$
4. $F = [2z, x, -y]$ $C: r(t) = \cos t\,\vec{i} + \sin t\,\vec{j} + 2t\vec{k}$, $0 \le t \le 2\pi$
5. $F = [(x-y), (y-z), (z-x)]$, $C: r(t) = 2\cos t\,\vec{i} + t\vec{j} + 2\sin t\,\hat{k}$ $(0 \le t \le 2\pi)$
6. $F = [e^x, e^{-y}, e^z]$, $C: r(t) = [t, t^2, t]$ from $(0,0,0)$ to $(1,1,1)$
7. $F = [\cosh x, \sinh y, e^z]$, $C: r(t) = [t, t^2, t^3]$ from $(0,0,0)$ to $(2,4,8)$

Path Independence of Line Integrals

(PP-1)

Definition: A line integral $\int_C F\cdot dr = \int_C F_1 dx + F_2 dy + F_3 dz$ is said to be independent of path in a domain D in space for every pair of end points A & B in D if the line integral has the same value for all paths in D that begin at A and end at B.

Theorem A line Integral: $\int_C F\cdot dr = \int_C F_1 dx + F_2 dy + F_3 dz$ with contineous F_1, F_2 & F_3 in a domain D in space is independent of path in D iff only if $F = [F_1, F_2, F_3]$ is the gradient of some function in D. i.e. $F = \text{grad} f$

In components: $F_1 = \frac{\partial f}{\partial x}$, $F_2 = \frac{\partial f}{\partial y}$ & $F_3 = \frac{\partial f}{\partial z}$

Example: Show that the line integral $\int_C F\cdot dr = \int F_1 dx + F_2 dy + F_3 dz = \int 2x\,dx + 2y\,dy + 2z\,dz$ is independent of path & hence evaluate from $A(0,0,0)$ to $B(2,2,2)$.

sol^n Here $F = 2x\vec{i} + 2y\vec{j} + 2z\vec{k}$ which is the gradient of the function $f = x^2+y^2+z^2$ $\left(\because \text{grad} f = \frac{\partial f}{\partial x}\vec{i} + \frac{\partial f}{\partial y}\vec{j} + \frac{\partial f}{\partial z}\vec{k}\right)$

$\Rightarrow \frac{\partial f}{\partial x} = 2x, \frac{\partial f}{\partial y} = 2y, \frac{\partial f}{\partial z} = 2z$

Hence the no doubt the line integral $\int_C 2x\,dx + 2y\,dy + 2z\,dz$ is path independent.

As the line integral path independent, its value remains same irrespective of all paths from $A(0,0,0)$ to $B(2,2,2)$.

Lets choose the simplest path i.e. the straight line whose parametric form is

$C: r(t) = t\vec{i} + t\vec{j} + t\vec{k}$ $(0 \le t \le 2)$

$\therefore x = t, y = t, z = t$

$\Rightarrow F(r(t)) = 2t\vec{i} + 2t\vec{j} + 2t\vec{k}$.

$\frac{dr}{dt} = 2\vec{i} + 2\vec{j} + 2\vec{k}$

$\Rightarrow F(r(t))\cdot\frac{dr}{dt} = 4t + 4t + 4t = 12t$ $(\because a\cdot b = a_1a_2 + b_1b_2 + c_1c_2)$

$\therefore \int_C F\cdot dr = \int_C 2x\,dx + 2y\,dy + 2z\,dz = \int_0^2 12t\,dt = 12\left[\frac{t^2}{2}\right]_0^2$

$= 6\times 4 = 24$ Ans.

Theorem-2

The line Integral $\int_C \vec{F}\cdot d\vec{r} = \int_C F_1 dx + F_2 dy + F_3 dz$ is independent of path in a domain D if and only if its value around any closed path in D is zero.
(To remember)

Exactness and Path Independence

Defn The differential form under the integral sign of a line integral: $\int_C F\cdot dr = \int (F_1 dx + F_2 dy + F_3 dz)$ i.e. $F_1 dx + F_2 dy + F_3 dz$ is said to be exact if we can find a function $f(x,y,z)$ s.t. $\boxed{F_1 = \frac{\partial f}{\partial x},\ F_2 = \frac{\partial f}{\partial y}\ \&\ F_3 = \frac{\partial f}{\partial z}}$

i.e. $F_1 dx + F_2 dy + F_3 dz = df$ i.e. $F = \text{grad} f$.

Theorem-3 (criterion for exactness & Path Independence)

Let F_1, F_2 & F_3 in the line integral $\int_C F\cdot dr = \int_C F_1 dx + F_2 dy + F_3 dz$ be contineous and have contineous first order partial differential equations in a domain D in space. Then;

(a) If the line integral is independent of path in D & thus the differential form "$F_1 dx + F_2 dy + F_3 dz$" under the integral sign is exact in D, then $\boxed{\text{Curl } F = 0}$

i.e. $$\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ F_1 & F_2 & F_3 \end{vmatrix} = 0 \Rightarrow \left(\frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z}\right)\hat{i} + \left(\frac{\partial F_1}{\partial z} - \frac{\partial F_3}{\partial x}\right)\hat{j} + \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right)\hat{k} = 0$$

In components.

$$\boxed{\frac{\partial F_3}{\partial y} = \frac{\partial F_2}{\partial z},\ \frac{\partial F_1}{\partial z} = \frac{\partial F_3}{\partial x},\ \frac{\partial F_2}{\partial x} = \frac{\partial F_1}{\partial y}} \quad \text{Imp (Condn for exactness)} \quad (1)$$

(b) If Curl $F = 0$ in D & D is simply connected, then the line integral is independent of path in D. (Imp).

Note If the differential form is exact in a line integral $\int_C F\cdot dr = \int_C F_1 dx + F_2 dy + F_3 dz$, then the line integral is path independent. (i.e. its value remains same along all paths bet^n A to B)

Working procedure to test exactness & evaluation of the line Integral $\int_C \vec{F}\cdot d\vec{r} = \int_C F_1dx + F_2dy + F_3dz$

Step-1 Test the exactness of the differential form $F_1dx + F_2dy + F_3dz$ i.e. show if $\frac{\partial F_3}{\partial y} = \frac{\partial F_2}{\partial z}$, $\frac{\partial F_1}{\partial z} = \frac{\partial F_3}{\partial x}$, $\frac{\partial F_2}{\partial x} = \frac{\partial F_1}{\partial y}$, If satisfies then ~~is~~ the line integral is exact & proceed to step-2 orelse It is not exact & stop.

Step-2 Assume, a function 'f' & integrate to get 'f' from the ~~integral~~ eqn

$$\frac{\partial f}{\partial x} = F_1 \quad \text{(Assume } x \text{ is only variable whereas } y \text{ \& } z \text{ are constants)}$$

i.e. $f = \int F_1dx + h(y,z)$ — (1) ($\because$ y & z are consts & hence they may exist in integration constant)

& get the function f.

Step-3 Find $\frac{\partial f}{\partial y}$ by partially differentiating w.r.t. y & equate with F_2

$\therefore \frac{\partial f}{\partial y} = F_2 \Rightarrow \boxed{f = \int F_2dy + g(z)}$ — (2) (Treat z as const & y as variable)

Step-4 Find $\frac{\partial f}{\partial z}$ by partially differentiating w.r.t z & equate with F_3

i.e. $\frac{\partial f}{\partial z} = F_3 \Rightarrow \boxed{f = \int F_3dz + K}$ — (3)

Step-5 Now 'f' as obtained in eqn 3 is the soln of the given line integral & substitute the given A(initial pt) and B(terminal pt) so as to find the desired value of the ~~is~~ given line integral.

i.e. $\boxed{I = \int_C \vec{F}\cdot d\vec{r} = \int_A^B F_1dx + F_2dy + F_3dz = f(B) - f(A)}$ Ans.

Example Test the exactness of the differential form of the line integral: $\int_{(0,1,2)}^{(1,-1,7)} 3x^2dx + 2yz\,dy + y^2dz$ and hence evaluate it.

$\underline{\text{sol}^n}$ The given line integral is

$$\int_{(0,1,2)}^{(1,-1,7)} 3x^2dx + 2yz\,dy + y^2dz$$

Here $F_1 = 3x^2$, $F_2 = 2yz$, $F_3 = y^2$.

<u>Now for test of exactness</u>

$$\frac{\partial F_3}{\partial y} = 2y, \quad \frac{\partial F_2}{\partial z} = 2y \Rightarrow \frac{\partial F_3}{\partial y} = \frac{\partial F_2}{\partial z}$$

$$\frac{\partial F_1}{\partial z} = 0, \quad \frac{\partial F_3}{\partial x} = 0 \Rightarrow \frac{\partial F_1}{\partial z} = \frac{\partial F_3}{\partial x}$$ (step-1)

$$\frac{\partial F_2}{\partial x} = 0 \quad \frac{\partial F_1}{\partial y} = 0 \Rightarrow \frac{\partial F_2}{\partial x} = \frac{\partial F_1}{\partial y}$$

Hence the diff. form is exact.

Now Let $\frac{\partial f}{\partial x} = F_1$

$$\Rightarrow \frac{\partial f}{\partial x} = 3x^2$$

$$\Rightarrow f = \int 3x^2dx + g(y,z)$$ (step-2)

$$\Rightarrow \boxed{f = x^3 + g(y,z)} \quad \text{——(1)}$$

~~Let~~ Now from eq^n 1

$$\frac{\partial f}{\partial y} = \frac{\partial g}{\partial y} = F_2$$

$$\Rightarrow \frac{\partial g}{\partial y} = 2yz \Rightarrow g = \int 2yz\,dy + h(z)$$ (step-3)

$$\Rightarrow \boxed{g = y^2z + h(z)} \quad \text{——(2)}$$

Now from eq^n 2 & eq^n 1

$$\boxed{f = x^3 + y^2z + h(z)} \quad \text{——(3)}$$

Diff. partially w.r.t z.

$$\frac{\partial f}{\partial z} = y^2 + \frac{dh}{dz} = F_3 \Rightarrow \cancel{\frac{dh}{dz}}\; y^2 + \frac{dh}{dz} = y^2$$

$$\Rightarrow \frac{dh}{dz} = 0 \Rightarrow \boxed{h = c} \quad \text{——(4)}$$ (step-4)

Now from eq^n 3 & 4: $\boxed{f = x^3 + y^2z + c}$

As $f = x^3 + y^2 z + c$

$$\Rightarrow \int_{(0,1,2)}^{(1,-1,7)} 3x^2 dx + 2yz\,dy + y^2 dz = f(B) - f(A)$$

$$= f(1,-1,7) - f(0,1,2)$$

$$= (1^3 + (-1)^2 \times 7 + c) - (0^3 + 1^2 \times 2 + c) = 8 - 2 = 6 \text{ Ans}$$

Example-2

Evaluate the line integral:

$\int_{(0,0,1)}^{(4,1,2)} 3y\,dx + 3x\,dy + 2z\,dz$ & verify weather the differential form under the integral sign is exact or not.

Sol^n $F_1 = 3y$, $F_2 = 3x$, $F_3 = 2z$.

Now $\frac{\partial F_3}{\partial y} = 0$, $\frac{\partial F_2}{\partial z} = 0$ $\Rightarrow \frac{\partial F_3}{\partial y} = \frac{\partial F_2}{\partial z}$

$\frac{\partial F_1}{\partial z} = 0$, $\frac{\partial F_3}{\partial x} = 0$ $\Rightarrow \frac{\partial F_1}{\partial z} = \frac{\partial F_3}{\partial x}$

$\frac{\partial F_2}{\partial x} = 3$, $\frac{\partial F_1}{\partial y} = 3$ $\Rightarrow \frac{\partial F_2}{\partial x} = \frac{\partial F_1}{\partial y}$ $\Rightarrow$ Exact.

Now Let $\frac{\partial f}{\partial x} = F_1 = 3y$

$\Rightarrow f = \int 3y\,dx + g(y,z)$

$\Rightarrow \boxed{f = 3xy + g(y,z)}$ — (1)

From-1 $\frac{\partial f}{\partial y} = 3x + \frac{\partial g}{\partial y} = F_2$

$\Rightarrow 3x + \frac{\partial g}{\partial y} = 3x \Rightarrow \frac{\partial g}{\partial y} = 0$

$\Rightarrow g = \int 0\,dy + h(z)$

$\Rightarrow \boxed{g = h(z)}$ — (2)

From 1 & 2: $\boxed{f = 3xy + h(z)}$ — (3)

From-3 $\frac{\partial f}{\partial z} = F_3 \Rightarrow \frac{dh}{dz} = 2z$

$\Rightarrow h = \int 2z\,dz + c \Rightarrow \boxed{h = z^2 + c}$ — (4)

From 3 & 4 $\boxed{f = 3xy + z^2 + c}$

Now $\int_{(0,0,0)}^{(4,1,2)} 3y\,dx + 3x\,dy + 2z\,dz = f(4,1,2) - f(0,0,0)$

$$= (3 \times 4 \times 1 + 2^2 + c) - (0 + 0 + c) = 16 \text{ Ans.}$$

PP-6

Ex-3 Show that the diff. form under the integral sign of the line integral $\int_{(0,0,1)}^{(1,\frac{\pi}{4},2)} 2xyz^2dx + (x^2z^2 + z\cos yz)dy + (y\cos yz)dz$ is exact & hence evaluate it.

Sol^n Here $F_1 = 2xyz^2$, $F_2 = x^2z^2 + z\cos yz$, $F_3 = 2x^2yz + y\cos yz$

Now $\frac{\partial F_3}{\partial y} = 2x^2z - yz\sin yz$, $\frac{\partial F_2}{\partial z} = 2x^2z - yz\sin yz \Rightarrow \frac{\partial F_3}{\partial y} = \frac{\partial F_2}{\partial z}$ ✓

$\frac{\partial F_1}{\partial z} = 4xyz$, $\frac{\partial F_3}{\partial x} = 4xyz \Rightarrow \frac{\partial F_1}{\partial z} = \frac{\partial F_3}{\partial x}$ ✓

$\frac{\partial F_2}{\partial x} = 2xz^2$, $\frac{\partial F_1}{\partial y} = 2xz^2 \Rightarrow \frac{\partial F_2}{\partial x} = \frac{\partial F_1}{\partial y}$ ✓

Hence the form is exact.

Now let $\frac{\partial f}{\partial x} = F_1 = 2xyz^2$

$\Rightarrow f = \int 2xyz^2dx + g(y,z)$

$\Rightarrow \boxed{f = x^2yz^2 + g(y,z)}$ — (1)

From -1 $\frac{\partial f}{\partial y} = F_2 \Rightarrow \cancel{x^2z^2} + \frac{\partial g}{\partial y} = \cancel{x^2z^2} + z\cos yz$

$\Rightarrow \frac{\partial g}{\partial y} = z\cos yz \Rightarrow g = \int z\cos yz\, dy + h(z)$

$\Rightarrow \boxed{g = \sin yz + h(z)}$ — (2)

From 1 & 2

$\boxed{f = x^2yz^2 + \sin yz + h(z)}$ — (3)

Now from-3: $\frac{\partial f}{\partial z} = F_3$

$\Rightarrow \cancel{2x^2yz + y\cos yz} + \frac{dh}{dz} = \cancel{2x^2yz + y\cos yz}$

$\Rightarrow \frac{dh}{dz} = 0 \Rightarrow \boxed{h = c}$ — (4)

From eq^n 3 & 4, we have

$$\boxed{f = x^2yz^2 + \sin yz + c}$$

$\therefore I = \int_{(0,0,1)}^{(1,\frac{\pi}{4},2)} [2xyz^2dx + (x^2z^2 + z\cos yz)dy + (2x^2yz + y\cos yz)dz]$

$= f(1,\frac{\pi}{4},2) - f(0,0,1)$

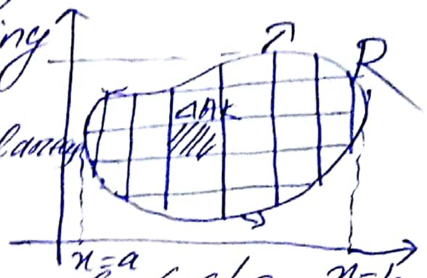
$= ((1^2 \times \frac{\pi}{4} \times 2^2 + \sin\frac{\pi}{2}) - (0 + \sin 0)) = \pi + 1$ Ans

Double Integrals

The definition and evaluation of Double Integrals are quite similar to the definite integrals (i.e. $\int_a^b f(x)dx$)

We subdivide the region R by drawing $\parallel^{el}$ lines to X-axis & Y-axis.

Then $J_n = \sum f(x_k, y_k)\Delta A_k$ (i.e summation of all areas (rectangles in R)



N.B All the properties of definite integrals ($\int_a^b f(x)dx$) are also preserved by the double integrals.

i.e. 1) $\iint_R K f(x,y)\,dx\,dy = K\iint_R f(x,y)\,dx\,dy$

2) $\iint_R (f+g)\,dx\,dy = \iint_R f\,dx\,dy + \iint_R g\,dx\,dy$

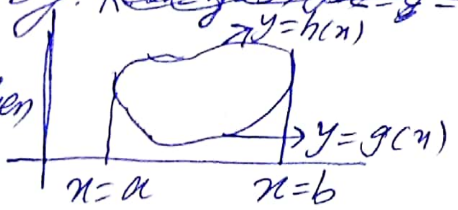
3) If $R = R_1 + R_2$, then $\iint_R f(x,y)\,dx\,dy = \iint_{R_1} f(x,y)\,dx\,dy + \iint_{R_2} f(x,y)\,dx\,dy$

Evaluation of Double Integrals

The double integrals are evaluated by two successive integrations as per the following cases.

Case-1: If the region R is described by: R ~~a≤x≤b, g(x)≤y~~ ≤

R: $a \le x \le b,\ g(x) \le y \le h(x)$, then



$$\iint_R f(x,y)\,dx\,dy = \int_a^b\left[\int_{g(x)}^{h(x)} f(x,y)\,dy\right]dx$$

(i.e. $f(x,y)$ first is to be integrated w.r.t. 'y' treating 'x' as a constant and then the resulting function (of x only) is to be integrated w.r.t 'x' over $a \le x \le b$)

Example Evaluate $\int_0^3\int_{-x}^{x}(x^2+y^2)\,dy\,dx$.

Sol^n
$$\int_0^3\int_{-x}^{x} x^2+y^2\,dy\,dx = \int_0^3\left[\int_{-x}^{x}(x^2+y^2)\,dy\right]dx \quad \text{(treat } x \text{ as const)}$$

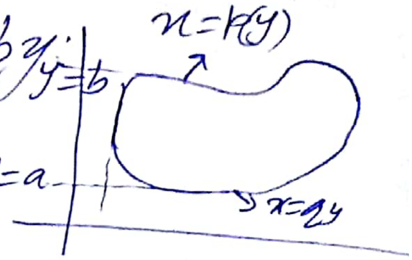
$$= \int_0^3\left(x^2[y]_{-x}^{x} + \left[\frac{y^3}{3}\right]_{-x}^{x}\right)dx$$

$$= \int_0^3\left(x^2(2x) + \frac{1}{3}(x^3+x^3)\right)dx = \int_0^3\left(2x^3 + \frac{2}{3}x^3\right)dx$$

$$= \int_0^3 \left(2+\frac{2}{3}\right) x^3 dx = \frac{8}{3}\int_0^3 x^3 dx$$

$$= \frac{8}{3}\left[\frac{x^4}{4}\right]_0^3 = \frac{2}{3}(81-0) = \frac{162}{3} \text{ Ans.}$$

Case-2 If the region R is described by:



R: $a \le y \le b$ & $p(y) \le x \le q(y)$, then

$$\iint_R f(x,y)\,dx\,dy = \int_a^b \left[\int_{p(y)}^{q(y)} f(x,y)\,dx\right] dy$$

→ (treat y as const.)

Example

Evaluate → $\int_0^{\frac{\pi}{4}} \int_0^y \frac{\sin y}{y}\, dx\,dy$

Solution $\int_0^{\frac{\pi}{4}}\int_0^y \frac{\sin y}{y}\,dx\,dy = \int_0^{\frac{\pi}{4}}\left[\int_0^y \frac{\sin y}{y}\,dx\right] dy$ (treat y-as const)

$$= \int_0^{\frac{\pi}{4}} \frac{\sin y}{y}[x]_0^y\,dy = \int_0^{\frac{\pi}{4}} \frac{\sin y}{y}(y-0)$$

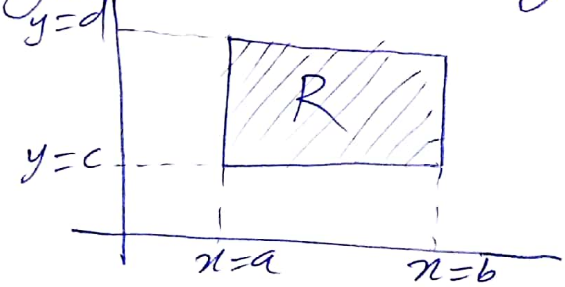
$$= \int_0^{\frac{\pi}{4}} \sin y\,dy = -[\cos y]_0^{\frac{\pi}{4}} = -\left(\frac{1}{\sqrt{2}}-1\right) = \left(1-\frac{1}{\sqrt{2}}\right) \text{ Ans}$$

Case-3 If the region is a rectangle and is defined by

R: $a \le x \le b$, $c \le x \le d$, then

$$\iint_R f(x,y)\,dx\,dy = \int_c^d\left[\int_a^b f(x,y)\,dx\right] dy$$

(treat-y as const)



Example

Evaluate: $\int_0^2\int_0^4 (x^2+y^2)\,dx\,dy$

solution $\int_0^2\left[\int_0^4 (x^2+y^2)\,dx\right] dy = \int_0^2\left(\left[\frac{x^3}{3}\right]_0^4 + y^2[x]_0^4\right) dy$

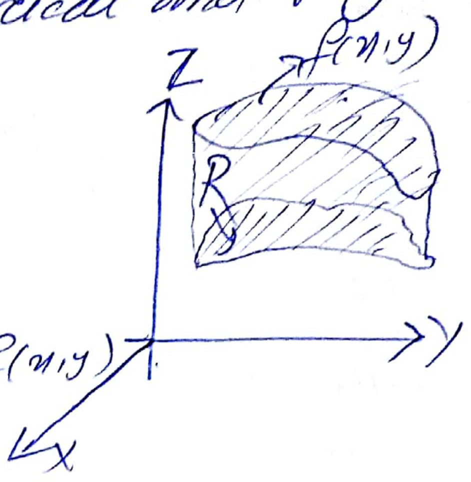
(y- is const)

$$= \int_0^2\left(\frac{64}{3}+4y^2\right)dy = \frac{64}{3}[y]_0^2 + \frac{4}{3}[y^3]_0^2$$

$$= \frac{128}{3}+\frac{32}{3} = \frac{160}{3} \text{ Ans.}$$

Applications of Double Integrals

Double Integrals have various geometrical and physical applications as per the followings.



① Area of a region 'R' in xy-plane:

$$\boxed{A = \iint_R dx\,dy}$$

② Volume (V) beneath the surface $z = f(x,y)$ and above a region R in xy-plane

$$\boxed{V = \iint_R f(x,y)\,dx\,dy}$$

③ Let $f(x,y)$ be the density (mass per unit area) of a distribution of mass in xy-plane, then the total mass is

$$\boxed{m = \iint_R f(x,y)\,dx\,dy}$$

④ Center of gravity $(\bar{x}, \bar{y})$ in co-ordinates of a mass (m)

$$\boxed{\bar{x} = \frac{1}{m}\iint_R x f(x,y)\,dx\,dy \;\&\; \bar{y} = \frac{1}{m}\iint_R y f(x,y)\,dx\,dy}$$

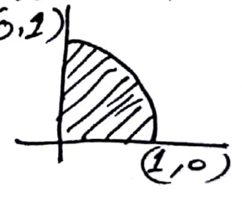
⑤ Moment of inertia: I_x & I_y of a mass in 'R' about x-axis and y-axis is

$$\boxed{I_x = \iint_R y^2 f(x,y)\,dx\,dy,\quad I_y = \iint_R x^2 f(x,y)\,dx\,dy}$$

Polar moment of inertia $\boxed{I_0 = I_x + I_y = \iint_R (x^2+y^2) f(x,y)\,dx\,dy}$

Examples

Q. Let $f(x,y) = 1$ be the density of a mass distribution in the region $R: 0 \le y \le \sqrt{1-x^2},\ 0 \le x \le 1$. Then find the C.G, & moment of inertia I_x, I_y & I_0.



Solution The total mass (m) in R is given by

$$m = \iint_R dx\,dy = \int_0^1\left[\int_0^{\sqrt{1-x^2}} dy\right]dx = \int_0^1 [y]_0^{\sqrt{1-x^2}}\,dx = \int_0^1 \sqrt{1-x^2}\,dx$$

Let $x = \sin\theta \Rightarrow dx = \cos\theta\, d\theta$, $0 \le \theta \le \frac{\pi}{2}$

$$\Rightarrow \int_0^1 \sqrt{1-x^2}\,dx = \int_0^{\frac{\pi}{2}} \cos^2\theta\, d\theta = \int \frac{1+\cos 2\theta}{2}\,d\theta = \frac{1}{2}\left\{[\theta]_0^{\frac{\pi}{2}} + \left[\frac{\sin 2\theta}{2}\right]_0^{\frac{\pi}{2}}\right\} = \frac{\pi}{4}$$

Now for center of gravity $(\bar{x}, \bar{y})$ $\boxed{f(x,y)=1}$ given

$\bar{x} = \frac{1}{m}\iint_R x f(x,y)\,dx\,dy = \frac{1}{(\frac{\pi}{4})}\iint_R x \cdot 1\,dx\,dy$

$= \frac{4}{\pi}\int_0^1 \left[x\int_0^{\sqrt{1-x^2}} dy\right]dx = \frac{4}{\pi}\int_0^1 x\sqrt{1-x^2}\,dx = -\frac{4}{\pi}\int_1^0 z^2\,dz$

(take $1-x^2 = z$)

$= \frac{4}{3\pi}$

Due to symmetry $\bar{y} = \frac{4}{3\pi}$

Hence $C \cdot G = \left(\frac{4}{3\pi}, \frac{4}{3\pi}\right)$ Ans

For moment of inertia

$I_x = \iint_R y^2 f(x,y)\,dx\,dy$, $I_y = \iint_R x^2 f(x,y)\,dx\,dy$.

$\therefore I_x = \iint_R y^2\,dx\,dy = \int_0^1\left[\int_0^{\sqrt{1-x^2}} y^2\,dy\right]dx = \int_0^1 \left[\frac{y^3}{3}\right]_0^{\sqrt{1-x^2}} dx$

$= \frac{1}{3}\int_0^1 (\sqrt{1-x^2})^3\,dx = \frac{1}{3}\int_0^{\frac{\pi}{2}} \cos^4\theta\,d\theta$ (take $x = \sin\theta$ $\Rightarrow dx = \cos\theta\,d\theta$ $(\sqrt{1-x^2})^3 = \cos^3\theta$)

$= \frac{1}{3}\left(\frac{3\pi}{16}\right) = \frac{\pi}{16}$

$I_y = \iint_R x^2 f(x,y)\,dx\,dy = \frac{\pi}{16}$ (due to symmetry)

$\therefore I_0 = I_x + I_y = \frac{\pi}{16} + \frac{\pi}{16} = \frac{2\pi}{16} = \frac{\pi}{8}$ Ans

Assignments

Exercise - 9-3

Green's Theorem in the Plane

(Transformation between double Integral & Line Integral)

Th^m

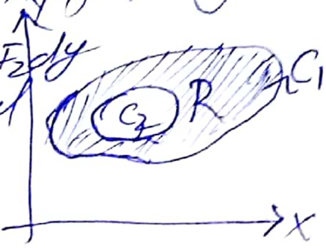
Let 'R' be a closed bounded region in xy-plane whose boundary 'C' consists of finitely many smooth curves. Let $F_1(x,y)$ and $F_2(x,y)$ be functions that are contineous and have contineous partial derivatives $\frac{\partial F_1}{\partial y}$ & $\frac{\partial F_2}{\partial x}$ every-where in some domain containing R. Then

$$\boxed{\iint_R \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right) dx\,dy = \oint_C (F_1\,dx + F_2\,dy)}$$

$\left(\text{i.e. } \int F \cdot dr = \iint (\text{curl}\,F)\cdot \hat{k}\, dx\,dy\right)$

Here we integrate along the entire boundary 'C' of 'R' s.t. 'R' is on the left as we advance in the direction of integration

(Here the line integral $\oint F \cdot dr = \oint_C F_1 dx + F_2 dy$ can be evaluated by the double integral $\iint_R (\text{curl}\,F)\cdot \hat{k}\, dx\,dy$)



Verification (Through Example)

Verify Green's Theorem in plane for $F = (y^2 - 7y)\vec{i} + (2xy + 2x)\vec{j}$ and $C: x^2 + y^2 = 1$ (circle)

Pf Double Integral (LHS)

$\iint_R \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right) dx\,dy$, Here $F_1 = y^2 - 7y$, $F_2 = 2xy + 2x$

Now $\frac{\partial F_2}{\partial x} = \frac{\partial}{\partial x}(2xy + 2x) = 2y + 2$

$\frac{\partial F_1}{\partial y} = \frac{\partial}{\partial y}(y^2 - 7y) = 2y - 7$

Now $\iint_R \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right) dx\,dy = \iint_R (2y+2) - (2y-7)\, dx\,dy$

$= \iint_R 9\,dx\,dy = 9\left(\iint_R dx\,dy\right) = 9 \times \text{Area of } R = 9(\pi \times 1^2) = 9\pi$

$\left(\because A = \iint_R dx\,dy\right)$

Line Integral (RHS)

Here $F = (y^2 - 7y)\vec{i} + (2xy + 2x)\vec{j}$

$C: r(t) = \cos t\,\vec{i} + \sin t\,\vec{j}$ $(0 \le t \le 2\pi)$ ($C: x^2 + y^2 = 1$, circle with radius $= 1$)

$\Rightarrow x = \cos t,\ y = \sin t.$

Now $F(r(t)) = (\sin^2 t - 7\sin t)\hat{i} + (2\sin t\cdot\cos t + 2\cos t)\hat{j}$

$\frac{dr}{dt} = -\sin t\,\hat{i} + \cos t\,\hat{j}$

Now $F(r(t))\cdot\frac{dr}{dt} = -\sin^3 t + 7\sin^2 t + 2\sin t\cdot\cos^2 t + 2\cos^2 t$

$= -\sin^3 t + 5\sin^2 t + 2\sin t\cdot\cos^2 t + 2$

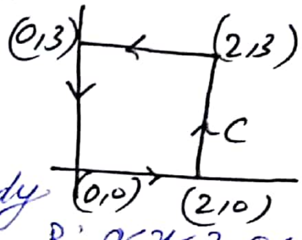
$\therefore \oint_C F_1dx + F_2dy = \int_0^{2\pi}(\sin^3 t + 5\sin^2 t + 2\sin t\cdot\cos^2 t + 2)\,dt$

$= 0 + 7\pi + 0 + 2\pi = 9\pi$

Thus LHS = RHS $\Rightarrow$ Green's Th^m is verified.

Evaluation of Line Integrals by Green's Th^m

Q. Using Green's th^m evaluate $\oint_C F\cdot dr$ counterclockwise around the boundary of C defined as the rectangle with vertices (0,0), (2,0), (2,3), (0,3) where $F = [x^2e^y, y^2e^x] = x^2e^y\hat{i} + y^2e^x\hat{j}$



Sol^n $F_1 = x^2e^x$, $F_2 = y^2e^x$

Now $\frac{\partial F_2}{\partial x} = y^2e^x$, $\frac{\partial F_1}{\partial y} = x^2e^y$

Now As per Green's Th^m $\oint_C F\cdot dr = \iint_R\left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right)dxdy$ R: $0\le x\le 2,\ 0\le y\le 3$

$= \int_0^3\int_0^2 (y^2e^x - x^2e^y)\,dxdy = \int_0^3\left[\int_0^2 (y^2e^x - x^2e^y)\,dx\right]dy$

$= \int_0^3\left[y^2[e^x]_0^2 - e^y\left[\frac{x^3}{3}\right]_0^2\right]dy$

$= \int_0^3\left(y^2(e^2-1) - \frac{8}{3}e^y\right)dy = (e^2-1)\left[\frac{y^3}{3}\right]_0^3 - \frac{8}{3}[e^y]_0^3$

$= \frac{e^2-1}{3}(27-0) - \frac{8}{3}(e^3-1) = 9(e^2-1) - \frac{8}{3}(e^3-1)$ Ans.

N.B The normal derivative of R of a sol^n $w(x,y)$ is denoted by $\frac{\partial w}{\partial n}$ and the Green's Th^m to find the integral of the normal derivative $\oint \frac{\partial w}{\partial n}ds$ counterclockwise over 'C' can be evaluated by the formula.

$$\boxed{\oint_C \frac{\partial w}{\partial n}ds = \iint_R\left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right)ds = \iint_R \nabla^2 w\,dxdy = \iint_R\left(\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2}\right)dxdy.}$$

Ex Evaluate $\oint_C \frac{\partial w}{\partial n}ds$, $w = e^x + e^y$ R: Rectangle $0\le x\le 2,\ 0\le y\le 1$.

Sol^n $w = e^x + e^y \Rightarrow \frac{\partial^2 w}{\partial x^2} = e^x$, $\frac{\partial^2 w}{\partial y^2} = e^y$

Now $\oint_C \frac{\partial w}{\partial n}ds = \int_0^1\int_0^2\left(\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2}\right)dxdy = \int_0^1\left(\int_0^2 (e^x + e^y)\,dx\right)dy$

$= \int_0^1\left([e^x]_0^2 + e^y[x]_0^2\right)dy = \int_0^1\left((e^2-1) + 2e^y\right)dy = (e^2-1)[y]_0^1 + 2[e^y]_0^1$

$= (e^2-1) + 2(e-1)$ Ans.

Assignments: Problemset 9.4 —o—

Surfaces and Surface Normals

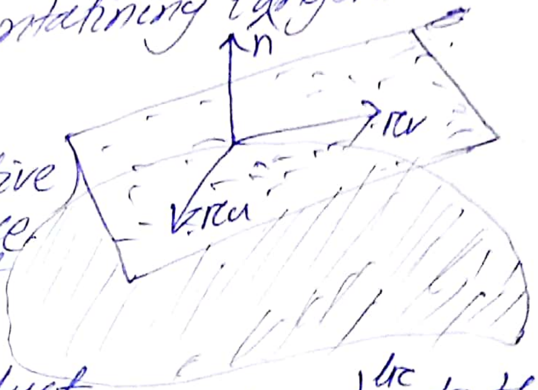
Parametric Representation of surfaces:

Name of surface	Parametric representation
General cartesian form $z=f(x,y)$ or $f(x,y)=0$	$\rightarrow r(u,v) = x(u,v)\hat{i} + y(u,v)\hat{j} + z(u,v)\hat{k}$
1. Sphere ($x^2+y^2+z^2=a^2$)	$\rightarrow r(u,v) = a\cos u \sin v\,\hat{i} + a\sin u \sin v\,\hat{j} + a\cos v\,\hat{k}$
2. Cylinder ($x^2+y^2=a^2, -1\le z\le 1$)	$\rightarrow r(u,v) = a\cos u\,\hat{i} + a\sin u\,\hat{j} + v\hat{k}$
3. XY-plane ($z=0$)	$\rightarrow r(u,v) = u\hat{i} + v\hat{j}$
4. cone	$\rightarrow r(u,v) = u\cos v\,\hat{i} + u\sin v\,\hat{j} + u^2\hat{k}$
5. Paraboloid of Revolution	$\rightarrow r(u,v) = u\cos v\,\hat{i} + u\sin v\,\hat{j} + u^2\hat{k}$
6. Elliptic Paraboloid	$\rightarrow r(u,v) = au\cos v\,\hat{i} + bu\sin v\,\hat{j} + c\sin v\,\hat{k}$
7. Ellipsoid	$\rightarrow r(u,v) = a\cos v\cos u\,\hat{i} + b\cos v\sin u\,\hat{j} + c\cosh u\,\hat{k}$
8. Hyperboloid	$\rightarrow r(u,v) = a\sinh u\cos v\,\hat{i} + b\sinh u\sin v\,\hat{j} + v\hat{k}$
9. Helicoid	$\rightarrow r(u,v) = u\cos v\,\hat{i} + u\sin v\,\hat{j} + u\hat{k}$
10. Elliptic cylinder	$\rightarrow r(u,v) = a\cos v\,\hat{i} + b\sin v\,\hat{j} + u\hat{k}$

Surface Normals

Def^n A normal vector of a surface 'S' at a point 'P' is a vector perpendicular to the tangent plane of 'S' at P. .

N.B (Tangent plane means the plane containing tangent vectors i.e. $r_u\left(\frac{\partial r}{\partial u}\right)$, $r_v\left(\frac{\partial r}{\partial v}\right)$)



As all the derivatives (first derivative represents the tangent vectors, hence $\frac{\partial r}{\partial u}$ (r_u) & $\frac{\partial r}{\partial v}$ (r_v) are the tangent vectors.

We also know that the cross product of two vector $\vec{a}$ & $\vec{b}$ ($\vec{a}\times\vec{b}$) is in the direction $\perp^{ar}$ to the plane containg $\vec{a}$ & $\vec{b}$

Hence $\frac{\partial r}{\partial u}\times\frac{\partial r}{\partial v}$ ($r_u\times r_v$) is a vector perpendicular to the plane containing r_u & r_v i.e. tangent plane and hence '$r_u \times r_v$' represents the surface normal vector.

$\therefore$ $\boxed{N = r_u \times r_v}$ (surface Normal)

$\therefore$ Unit Normal vector $= \boxed{\hat{n} = \frac{\vec{r_u}\times\vec{r_v}}{|r_u \times r_v|}}$ Imp

Also from application of gradient, we know that gradient of a surface $f(x,y,z)=0$ represents the surface normal vector.

Hence the unit normal vector of a surface $f(x,y,z)=0$ is

PP-2

$$\boxed{\hat{n} = \frac{\text{grad} f}{|\text{grad} f|}}$$

Q1 Find the surface unit normal vector of the followings

1. XY-Plane : $r(u,v) = u\vec{i} + v\vec{j}$
2. Cone : $r(u,v) = u\cos v\vec{i} + u\sin v\vec{j} + cu\hat{k}$

Soln 1. $r(u,v) = u\vec{i} + v\vec{j}$

$\Rightarrow r_u = \frac{\partial r}{\partial u} = \frac{\partial}{\partial u}(u\vec{i} + v\vec{j}) = \vec{i}$

$r_v = \frac{\partial r}{\partial v} = \frac{\partial}{\partial v}(u\vec{i} + v\vec{j}) = \vec{j}$

$\therefore N = r_u \times r_v = \vec{i} \times \vec{j} = \hat{k}$ Ans

Unit normal vector $\hat{n} = \frac{\hat{k}}{|\hat{k}|} = \frac{\hat{k}}{1} = \hat{k}$ Ans

Soln 2 $r(u,v) = u\cos v\vec{i} + u\sin v\vec{j} + cu\vec{k}$

$\Rightarrow r_u = \frac{\partial r}{\partial u} = \cos v\vec{i} + \sin v\vec{j} + c\vec{k}$

$r_v = \frac{\partial r}{\partial v} = -u\sin v\vec{i} + u\cos v\vec{j}$

Now surface Normal $= N = r_u \times r_v = \begin{vmatrix} \vec{i} & \vec{j} & \hat{k} \\ \cos v & \sin v & c \\ -u\sin v & u\cos v & 0 \end{vmatrix}$

$\therefore N = -cu\cos v\vec{i} + (-cu\sin v)\vec{j} + u\underbrace{(\cos^2 v + \sin^2 v)}_{1}\vec{k}$

$\therefore \hat{n} = \frac{-cu\cos v\vec{i} - cu\sin v\vec{j} + u\vec{k}}{\sqrt{c^2u^2(\cos^2 v + \sin^2) + u^2}} = \frac{1}{u\sqrt{c^2+1}}(-cu\cos v\vec{i} - cu\sin v\vec{j} + u\vec{k})$

Q Find the surface and unit normal vector of the cartecian surface.

$4x^2 + y^2 + 9z^2 = 36$.

Soln Surface Normal = grad f , $f = 4x^2 + y^2 + 9z^2 - 36$

$N = \text{grad} f = \frac{\partial f}{\partial x}\vec{i} + \frac{\partial f}{\partial y}\vec{j} + \frac{\partial f}{\partial z}\vec{k}$, $\frac{\partial f}{\partial x} = 8x$, $\frac{\partial f}{\partial y} = 2y$, $\frac{\partial f}{\partial z} = 18z$

$\Rightarrow N = 8x\vec{i} + 2y\vec{j} + 18z\vec{k}$

Unit normal $= \hat{n} = \frac{\text{grad} f}{|\text{grad} f|} = \frac{8x\vec{i} + 2y\vec{j} + 18z\vec{k}}{\sqrt{(8x)^2 + (2y)^2 + (18z)^2}}$

$\therefore \hat{n} = \frac{8x\vec{i} + 2y\vec{j} + 18z\vec{k}}{\sqrt{64x^2 + 4y^2 + 324z^2}}$

Assignments

Surface Integrals

The surface integral of a vector function 'F' is defined as: $\boxed{\iint_S F\cdot\hat{n}\,dA = \iint_R F(r(u,v))\cdot N(u,v)\,du\,dv}$

where $N(u,v)$ is the surface normal vector ($r_u \times r_v$).

This is also called the flux integral.

Evaluation of Surface Integrals $\iint_S F\cdot\hat{n}\,dA$

step-1 Note the given vector function $F = F_1\hat{i} + F_2\hat{j} + F_3\hat{k}$ and write the parametric representation of the surface 'S' as $r(u,v) = x(u,v)\hat{i} + y(u,v)\hat{j} + z(u,v)\hat{k}$ and note x, y & z.

step-2 Substitute x, y & z of '$r(u,v)$' in F to get $F(r(u,v))$.

step-3 Find $r_u = \frac{\partial r}{\partial u}$, & $r_v = \frac{\partial r}{\partial v}$ & hence $r_u \times r_v$

$$\Rightarrow N = r_u \times r_v = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial x}{\partial u} & \frac{\partial y}{\partial u} & \frac{\partial z}{\partial u} \\ \frac{\partial x}{\partial v} & \frac{\partial y}{\partial v} & \frac{\partial z}{\partial v} \end{vmatrix}$$

step-4 Find the limits for u & v on S. $a \le u \le b$, $c \le v \le d$

step-5 Find the dot product of $F(r(u,v))$ & $N(u,v)$ i.e. $F(r(u,v))\cdot N(u,v)$

step-6 Integrate to get the surface integral $\iint_S F\cdot\hat{n}\,dA$

$$\iint_S F\cdot\hat{n}\,dA = \int_c^d\int_a^b \left(F(r(u,v))\cdot N(u,v)\right)du\,dv.$$ & evaluate the double integral to get the desired surface Integral.

Q Compute the flux of water through the parabolic cylinder $S: y = x^2, 0 \le x \le 2, 0 \le z \le 3$ if the velocity vector $v = [3z^2, 6, 6xz]$ ($\rho = 1\,gm/cm^3$)

Sol^n To get the flux of water (total mass flow/sec) through the pipe, we have to compute the surface integral $\iint_S v\cdot\hat{n}\,dA$.

(PP-4)

Here $F = V = [3z^2, 6, 6xz] = 3z^2\hat{i} + 6\hat{j} + 6xz\hat{k}$

Surface $S: y = x^2,\ 0 \le x \le 2,\ 0 \le z \le 3.$

Let $x = u,\ z = v,$

$\Rightarrow y = u^2$

Thus parametric form of 'S' is

$S: \boxed{r(u,v) = u\hat{i} + u^2\hat{j} + v\hat{k}}$ $\begin{matrix} 0 \le u \le 2 \\ 0 \le v \le 3 \end{matrix}$ (step-1)

$\Rightarrow x = u,\ y = u^2,\ z = v.$

(step-2)

Now $F = 3z^2\hat{i} + 6\hat{j} + 6xz\hat{k}$

$\Rightarrow F(r(u,v)) = 3v^2\hat{i} + 6\hat{j} + 6uv\hat{k}$ $(\because x = u, y = u^2, z = v)$

Here $r(u,v) = u\hat{i} + u^2\hat{j} + v\hat{k}$

$\Rightarrow r_u = \frac{\partial r}{\partial u} = \hat{i} + 2u\hat{j}$

$r_v = \frac{\partial r}{\partial v} = \hat{k}$

Now $N = r_u \times r_v = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 2u & 0 \\ 0 & 0 & 1 \end{vmatrix} = 2u\hat{i} - \hat{j}$ (step-3)

$\therefore \boxed{N(u,v) = 2u\hat{i} - \hat{j}}$

As per def^n of S: $0 \le u \le 2,\ 0 \le v \le 3$ (step-4)

Now $F(r(u,v)) \cdot N(u,v)$

$= (3v^2\hat{i} + 6\hat{j} + 6uv\hat{k}) \cdot (2u\hat{i} - \hat{j})$ (step-5)

$\Rightarrow \boxed{F(r(u,v)) \cdot (N(u,v)) = 6uv^2 - 6}$ $(\because a \cdot b = a_1a_2 + b_1b_2 + c_1c_2)$

Now total flux $= \iint_S F \cdot \hat{n}\, dA = \iint_R F\, r(u,v) \cdot N(u,v)\, du\, dv$

$= \int_0^3 \int_0^2 (6uv^2 - 6)\, du\, dv. = \int_0^3 \left[\int_0^2 (6uv^2 - 6)\, du \right] dv$ (treat v as const)

$= \int_0^3 \left(6v^2 \left[\frac{u^2}{2} \right]_0^2 - 6 [u]_0^2 \right) dv = \int_0^3 (12v^2 - 12)\, dv$

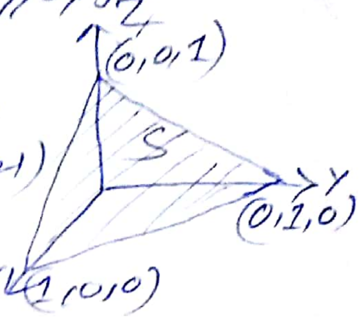
$= 12 \left[\frac{v^3}{3} \right]_0^3 - 12 [v]_0^3 = 4 \times (27 - 0) - 12 \times (3 - 0)$

$= 108 - 36 = 72$ (m^3/sec).

Example-2:

Evaluate the surface integral $\iint_S F\cdot\hat{n}\,dA$, where $F=[x^2, e^y, 1]=x^2\hat{i}+e^y\hat{j}+\hat{k}$, $S: x+y+z=1,\ x\geq 0, y\geq 0, z\geq 0$

Sol^n Here $F=x^2\hat{i}+e^y\hat{j}+\hat{k}$, $S: x+y+z=1,\ x\geq 0, y\geq 0, z\geq 0$



__Parametric representation of 'S'__

$x+y+z=1$, let $x=u,\ y=v \Rightarrow z=1-u-v$

$\therefore S: r(u,v)=u\hat{i}+v\hat{j}+(1-u-v)\hat{k}$. $\begin{matrix}0\leq u\leq 1\\ 0\leq v\leq 1\end{matrix}$ (step-1)

Now $x=u,\ y=v,\ z=1-u-v$ (step-2)

$\Rightarrow \boxed{F(r(u,v))=u^2\hat{i}+e^v\hat{j}+\hat{k}}$

Now $r(u,v)=u\hat{i}+v\hat{j}+(1-u-v)\hat{k}$

$\Rightarrow r_u=\frac{\partial r}{\partial u}=\hat{i}-\hat{k}$

$r_v=\frac{\partial r}{\partial v}=\hat{j}-\hat{k}$ (step-3)

$\Rightarrow N(u,v)=r_u\times r_v=\begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ 1 & 0 & -1\\ 0 & 1 & -1\end{vmatrix}=\hat{i}+\hat{j}+\hat{k}$

Here $0\leq u\leq 1,\ 0\leq v\leq 1$ (step-4)

Now $F(r(u,v))\cdot N(u,v)=(u^2\hat{i}+e^v\hat{j}+\hat{k})\cdot(\hat{i}+\hat{j}+\hat{k})$

$=u^2+e^v+1$

$\Rightarrow F(r(u,v))\cdot N(u,v)=u^2+e^v+1$ (step-5)

Hence $\iint_S F\cdot\hat{n}\,dA=\iint_R F(r(u,v))\cdot N(u,v)\,du\,dv$ (step-6)

$=\int_0^1\int_0^1(u^2+e^v+1)\,du\,dv=\int_0^1\left[\int_0^1(u^2+e^v+1)\,du\right]dv$ (treat v as const)

$=\int_0^1\left(\left[\frac{u^3}{3}\right]_0^1+e^v[u]_0^1+[u]_0^1\right)dv=\int_0^1\left(\frac{1}{3}+e^v+1\right)dv$

$=\frac{1}{3}[v]_0^1+[e^v]_0^1+[v]_0^1=\frac{1}{3}+(e^1-1)+1=e+\frac{1}{3}$ Ans.

__Theorem (Change of orientation)__

The replacement of $\hat{n}$ by $-\hat{n}$ (or N by $-N$) in the surface integral $\iint F\cdot\hat{n}\,dA$ corresponds to the multiplication of the integral by (-1).

Applications of Surface Integral

Another type of surface integral is

$$\boxed{\iint_S G(\vec{r})\,dA = \iint_R G(\vec{r}(u,v))\,|N(u,v)|\,du\,dv}$$ which is

independent of orientation.

Here $dA = |N|\,du\,dv = |\vec{r}_u \times \vec{r}_v|\,du\,dv$.

Hence the total mass density of 'S' $(G=1)$ gives the area. Thus Area of $\boxed{S = A(S) = \iint_S dA = \iint_R |\vec{r}_u \times \vec{r}_v|\,du\,dv.}$

Ex Find the area of the sphere $x^2+y^2+z^2=a^2$.

Soln Here parametric form of S: $\vec{r}(u,v) = a\cos u\cos v\,\hat{i} + a\sin u\cos v\,\hat{j} + a\sin v\,\hat{k}$

$0 \le u \le 2\pi,\ -\frac{\pi}{2} \le v \le \frac{\pi}{2}$

Now $|\vec{r}_u \times \vec{r}_v| = \left|\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ -a\sin u\cos v & a\cos u\cos v & 0 \\ -a\cos u\sin v & -a\sin u\sin v & a\cos v \end{vmatrix}\right| = a^2|\cos v|$

Now $A(S) = \iint_S dA = \iint_R |\vec{r}_u \times \vec{r}_v| = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \left[\int_0^{2\pi} a^2\cos v\,du\right] dv$

$= 2\pi a^2 \int_{-\pi/2}^{\pi/2} \cos v\,dv = 2\pi a^2 \left[\sin v\right]_{-\pi/2}^{\pi/2} = 2\pi a^2(1+1) = 4\pi a^2$

Moment of Inertia

$$\boxed{I = \iint_S \mu D^2\,dA}$$

$\mu = \mu(x,y,z)$ = mass distribution

$D(x,y,z)$ = distance of the point.

Another Representation of Surface integral

$$\boxed{\iint_S G(\vec{r})\,dA = \iint_{R^*} G(x,y,f(x,y))\sqrt{1+\left(\frac{\partial f}{\partial x}\right)^2+\left(\frac{\partial f}{\partial y}\right)^2}\,dx\,dy}$$

Here R^* is projection of 'S' in XY-plane.

$$\boxed{A_S = \iint_{R^*} \sqrt{1+\left(\frac{\partial f}{\partial x}\right)^2+\left(\frac{\partial f}{\partial y}\right)^2}}$$ Area of 'S'

Assignments

Triple Integrals and Gauss Divergence Theorem

The triple integral is a generalization of double integrals. Let $f(x,y,z)$ is a function defined in a closed & bounded region 'T' in space. We subdivide this 3-dimensional region 'T' by planes to the 3 co-ordinate planes. Then these boxes of subdivision that lie entirely inside 'T' are numbered 1 to n, then the triple integral is in the form of the sum

$$J_n = \sum_{k=1}^{n} f(x_k, y_k, z_k)\Delta V_k \ , \ \Delta V_k = \text{volume of } k^{th} \text{ box.}$$

i.e. $$\boxed{\iiint_T f(x,y,z)\,dx\,dy\,dz = \iiint_T f(x,y,z)\,dV = \sum_{k=1}^{n} f(x_k, y_k, z_k)\Delta V_k}$$

Here $\iiint f(x,y,z)\,dV$ or $\iiint_T f(x,y,z)\,dV$ is the representation of the triple integrals. It evaluation process is very similar as the double integrals.

Ex Evaluate the triple integral $\iiint_T (x^2+y^2+z^2)\,dx\,dy\,dz$, where $T: |x| \le 1, \ |y| \le 3, \ |z| \le 2$

<u>sol^n</u> Here $T: |x| \le 1, \ |y| \le 3, \ |z| \le 2$

$|x| \le 1 \Rightarrow -1 \le x \le 1 \ , \ |y| \le 3 \Rightarrow -3 \le y \le 3, \ |z| \le 2 \Rightarrow -2 \le z \le 2$

Now $$\iiint_T (x^2+y^2+z^2)\,dx\,dy\,dz = \int_{-2}^{2}\int_{-3}^{3}\int_{-1}^{1} (x^2+y^2+z^2)\,dx\,dy\,dz$$

$$= \int_{-2}^{2}\int_{-3}^{3}\left[\int_{-1}^{1}(x^2+y^2+z^2)\,dx\right]dy\,dz$$

(y & z const.)

$$= \int_{-2}^{2}\int_{-3}^{3}\left(\left[\frac{x^3}{3}\right]_{-1}^{1} + (y^2+z^2)\left[x\right]_{-1}^{1}\right)dy\,dz$$

$$= \int_{-2}^{2}\left[\int_{-3}^{3} \frac{2}{3} + 2(y^2+z^2)\,dy\right]dz$$

(treat z - as const)

$$= \int_{-2}^{2}\left(\frac{2}{3}\left[y\right]_{-3}^{3} + 2\left[\frac{y^3}{3}\right]_{-3}^{3} + z^2\left[y\right]_{-3}^{3}\right)dz$$

$$= \int_{-2}^{2}\left(\frac{2}{3}(6) + \frac{2}{3}\times 54 + 6z^2\right)dz = \int_{-2}^{2}\left((4+36)+6z^2\right)dz$$

$$= \int_{-2}^{2}(40+6z^2)\,dz = 40\left[z\right]_{-2}^{2} + \frac{6}{3}\left[z^3\right]_{-2}^{2}$$

$$= 40\times 4 + 2\times 16 = 160+32$$

$$= 192 \text{ Ans.}$$

The Divergence theorem of Gauss (PP-2)

(Transformation between volume integral and surface Integral)

Let T is a closed and bounded region in space whose boundary is a piecewise smooth orientable surface S. Let $F(x,y,z)$ be a vector function that is continuous and has cts first order partial derivative in some domain containing T. Then

$$\boxed{\iiint_T \operatorname{div} F \, dV = \iint_S F\cdot\hat{n}\, dA.}$$

$$\boxed{\operatorname{div} F = \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z}}$$

Here $\hat{n}$ is the outer unit normal vector of 'S'.

Example: Using Gauss Divergence theorem, evaluat the following surface integral.

$$\iint_S x^3\,dy\,dz + x^2y\,dz\,dx + x^2z\,dx\,dy,$$

$\boxed{S: \text{closed surface consisting of the cylinder } x^2+y^2=a^2\ (0\le z\le b)\ \&\ \text{circular disk } z=0\ \&\ z=b\ (x^2+y^2\le a^2)}$

Sol^n We have to evalute $\iint_S F\cdot\hat{n}\,dA$, where

$F = x^3\hat{i} + x^2y\hat{j} + x^2z\hat{k}$, i.e. $F_1 = x^3$, $F_2 = x^2y$, $F_3 = x^2z$

Now $\operatorname{div} F = \frac{\partial F_1}{\partial x}\hat{i} + \frac{\partial F_2}{\partial y}\hat{j} + \frac{\partial F_3}{\partial z}\hat{k}$ $\quad \frac{\partial F_1}{\partial x} = 3x^2$, $\frac{\partial F_2}{\partial y} = x^2$, $\frac{\partial F_3}{\partial z} = x^2$

$\Rightarrow \boxed{\operatorname{div} F = 3x^2 + x^2 + x^2} \Rightarrow \boxed{\operatorname{div} F = 5x^2}$

As per Gauss Divergence theorem

$$\iint_S F\cdot\hat{n}\,dA = \iiint_T \operatorname{div} F\, dV$$

Introducing polar co-ordinate for the surface/region, we hv (Cylindrical co-ordinat r,θ,z)

$x = r\cos\theta$, $y = r\sin\theta$,

we have $dx\,dy\,dz = r\,dr\,d\theta\,dz$

Now $\iint_S F\cdot\hat{n}\,dA = \iiint_T 5x^2\,dx\,dy\,dz = 5\int_{z=0}^{b}\int_{r=0}^{a}\int_{\theta=0}^{2\pi} r^2\cos^2\theta\; r\,dr\,d\theta\,dz$

$= 5\int_{z=0}^{b}\int_{r=0}^{a}\left[\int_0^{2\pi} r^3\cos^2\theta\,d\theta\right]dr\,dz$ (r = const) $= 5\int_{z=0}^{b}\int_{r=0}^{a} r^3\left(\frac{1}{2}[\theta]_0^{2\pi} + \left[\frac{\sin 2\theta}{2}\right]_0^{2\pi}\right)dr\,dz$

$= 5\int_{z=0}^{b}\left[\int_0^a \pi r^3\,dr\right]dz = 5\int_0^b \frac{\pi}{4}\left[r^4\right]_0^a dz$

$= \frac{5\pi}{4}\times a^4\int_0^b dz = \frac{5\pi}{4}a^4[z]_0^b = \frac{5\pi}{4}a^4(b-0)$

$= \frac{5\pi}{4}a^4 b$ **Ans**

Verification through Example

Evaluate and verify the Gauss Divergence theorem of the following surface integral.

$$\iint_S (7x\hat{i} - z\hat{k})\cdot\hat{n}\,dA \text{ , over the surface } S: x^2+y^2+z^2=4 \text{ (sphere)}$$

Pf We know that the Gauss Divergence theorem is

$$\iiint_T \operatorname{div} F\,dv = \iint_S F\cdot\hat{n}\,dA.$$

Here $F = 7x\hat{i} - z\hat{k}$, $S: x^2+y^2+z^2=4$

LHS $F_1 = 7x,\ F_2 = 0,\ F_3 = -z$

$\Rightarrow \frac{\partial F_1}{\partial x} = 7,\ \frac{\partial F_2}{\partial y} = 0,\ \frac{\partial F_3}{\partial z} = -1$

Hence $\operatorname{div} F = \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z} = 7+0-1 = 6$

Hence $\iiint_T \operatorname{div} F\,dv = \iiint_T 6\,dv = 6\iiint_T dv = 6\times$ volume of the sphere.

$= 6\times\frac{4}{3}\pi\times 2^3 = \frac{6\times 4\times 8}{3}\pi$ $(\because$ radius $= r = 2)$

$= 64\pi$ LHS.

RHS The given surface is $S: x^2+y^2+z^2 = 2^2$

$\Rightarrow$ Parametric representation of 'S' is $\left(\begin{matrix} 0\le u\le 2\pi \\ -\frac{\pi}{2}\le v\le\frac{\pi}{2}\end{matrix}\right)$

$S: r(u,v) = 2\cos u\cos v\,\hat{i} + 2\sin u\cos v\,\hat{j} + 2\sin v\,\hat{k}$

$\Rightarrow \frac{\partial r}{\partial u} = r_u = -2\sin u\cos v\,\hat{i} + 2\cos u\cos v\,\hat{j}$

$\frac{\partial r}{\partial v} = r_v = -2\cos u\sin v\,\hat{i} - 2\sin u\sin v\,\hat{j} + 2\cos v\,\hat{k}$

Now $N = r_u\times r_v = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ -2\sin u\cos v & 2\cos u\cos v & 0 \\ -2\cos u\sin v & -2\sin u\sin v & 2\cos v \end{vmatrix}$

$\Rightarrow N(u,v) = 4\cos u\cos^2 v\,\hat{i} + 4\sin u\cos^2 v + 4\sin v\cos v\,\hat{k}$

$F(r(u,v)) = 14\cos u\cos v\,\hat{i} - 2\sin v\,\hat{k}$.

$\Rightarrow F(r(u,v))\cdot N(u,v) = (14\cos u\cos v)(4\cos^2 v\cos u) + (-2\sin v)\,4\cos u\cos v$

$\Rightarrow F(r(u,v))\cdot N(u,v) = 56\cos^3 v\cos^2 u - 8\sin^2 v\cos v$

Hence $\iint_S F\cdot\hat{n}\,dA = \int_{-\pi/2}^{\pi/2}\left[\int_0^{2\pi}(56\cos^3 v\cos^2 u - 8\cos v\sin^2 v)\,du\right]dv$ treat v as const

$= \int_{-\pi/2}^{\pi/2}\left((56\cos^3 v)\pi - (8\cos v\sin^2 v)(2\pi)\right)dv$

$= 56\pi\left(2-\frac{2}{3}\right) - 16\pi\left(\frac{2}{3}\right) = 64\pi$ RHS

As LHS = RHS $\Rightarrow$ Gauss Divergence theorem is proved & value of the integral is 64π Ans.

Example

Evaluate $\iint_S F\cdot\hat{n}\,dA$ by using the divergence theorem of Gauss, where $F = e^x\hat{i} + e^y\hat{j} + e^z\hat{k}$ & $S: |x|\le 1, |y|\le 1, |z|\le 1$

PP-4

Soln Here $F = e^x\hat{i} + e^y\hat{j} + e^z\hat{k} \Rightarrow F_1 = e^x, F_2 = e^y, F_3 = e^z$

$\frac{\partial F_1}{\partial x} = e^x, \frac{\partial F_2}{\partial y} = e^y, \frac{\partial F_3}{\partial z} = e^z \Rightarrow \boxed{\text{grad}F = e^x + e^y + e^z}$

According to the divergence theorem of Gauss

$\iiint_T \text{div}F\,dV = \iint_S F\cdot\hat{n}\,dA$, $T: |x|\le 1, |y|\le 1, |z|\le 1$

$\Rightarrow -1\le x\le 1, -1\le y\le 1, -1\le z\le 1$

Hence $\iint_S F\cdot\hat{n}\,dA = \int_{-1}^{1}\int_{-1}^{1}\int_{-1}^{1} \text{div}F\cdot dx\,dy\,dz$

$= \int_{-1}^{1}\int_{-1}^{1}\left[\int_{-1}^{1}(e^x + e^y + e^z)\,dx\right]dy\,dz$ (y & z are const)

$= \int_{-1}^{1}\int_{-1}^{1}\left([e^x]_{-1}^{1} + e^y[x]_{-1}^{1} + e^z[x]_{-1}^{1}\right)dy\,dz$

$= \int_{-1}^{1}\int_{-1}^{1}\left((e-\frac{1}{e}) + 2e^y + 2e^z\right)dy\,dz = \int_{-1}^{1}\left[\int_{-1}^{1}((e-\frac{1}{e}) + 2e^y + 2e^z)\,dy\right]dz$ z = const

$= \int_{-1}^{1}\left((e-\frac{1}{e})[y]_{-1}^{1} + 2[e^y]_{-1}^{1} + 2e^z[y]_{-1}^{1}\right)dz$

$= \int_{-1}^{1}\left(2(e-\frac{1}{e}) + 2(e-\frac{1}{e}) + 4e^z\right)dz$

$= \int_{-1}^{1}\left((4e-\frac{4}{e}) + 4e^z\right)dz = (4e-\frac{4}{e})[z]_{-1}^{1} + 4[e^z]_{-1}^{1} = 8e - \frac{8}{e} + 4(e-\frac{1}{e})$

$= 12(e-\frac{1}{e})$ Ans.

Note If $\sigma = f(x,y,z)$ is the mass distribution in a region 'T' in space then the total mass is given by

$\boxed{\text{Total mass} = \iiint_T \sigma(x,y,z)\,dV}$

The moment of Inertia of a mass distribution ($f(x,y,z)=1$) is given by

$\boxed{I_x = \iiint_T (y^2+z^2)\,dx\,dy\,dz,\ I_y = \iiint_T (z^2+x^2)\,dx\,dy\,dz,\ I_z = \iiint_T (x^2+y^2)\,dx\,dy\,dz}$

The Divergence theorem of Gauss can be expressed in components as

$$\boxed{\iiint_T \text{div}F\,dx\,dy\,dz = \iint_S (F_1\cos\alpha + F_2\cos\beta + F_3\cos\gamma)\,dA \quad \text{Here } \hat{n} = \cos\alpha\,\hat{i} + \cos\beta\,\hat{j} + \cos\gamma\,\hat{k}}$$

Also

$$\boxed{\iiint_T \text{div}F\,dx\,dy\,dz = \iint_S F_1\,dy\,dz + F_2\,dz\,dx + F_3\,dx\,dy}$$

Assignments

Stokes Theorem

(Transformation between Surface Integral & Line Integral)

Statement

Let S be a piecewise smooth oriented surface in space and the boundary of 'S' be a piecewise smooth oriented simple closed curve 'C'. Let $F(x,y,z)$ be a contineous vector function that has contineous first partial derivatives in a domain in space containing S, Then

$$\boxed{\iint_S (\mathrm{Curl} F)\cdot \hat{n}\, dA = \oint_C F\cdot r'(s)\, ds}$$

where $\hat{n}$ = normal vector of 'S' and, depending on 'n', the integration around C is taken counterclockwise & $r' = \frac{dr}{ds}$ is unit tangent vector & s is length of C.

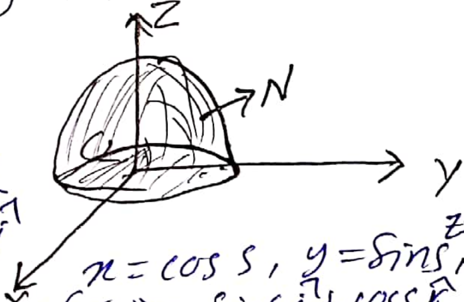
$$\boxed{N = r_u \times r_v = N_1\hat{i} + N_2\hat{j} + N_3\hat{k} = [N_1, N_2, N_3]}$$

In components

$$\boxed{\iint_R \left[\left(\frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z}\right)N_1 + \left(\frac{\partial F_1}{\partial z} - \frac{\partial F_3}{\partial x}\right)N_2 + \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right)N_3\right]du\,dv = \oint_C (F_1dx + F_2dy + F_3dz)}$$

Verification through Example

Verify stoke's theorem for the function $F = y\hat{i} + z\hat{j} + x\hat{k}$ & S is the paraboloid $z = f(x,y) = 1-(x^2+y^2),\ z \geq 0$



Pf RHS From the figure, it is clear that the curve 'C' is the circle $r(s) = \cos s\,\hat{i} + \sin s\,\hat{j}$

$x = \cos s,\ y = \sin s,\ z=0$

$F(r(s)) = \sin s\,\hat{i} + \cos s\,\hat{k}$

Its tangent vector is $\frac{dr}{ds} = -\sin s\,\hat{i} + \cos s\,\hat{j}$

$\Rightarrow$ Unit tangent vector $\hat{n} = \dfrac{-\sin s\,\hat{i} + \cos s\,\hat{j}}{\sqrt{\sin^2 s + \cos^2 s}} = -\sin s\,\hat{i} + \cos s\,\hat{j}$

Hence $\oint_C F\cdot dr = \oint_C \left(F(r(s))\cdot \frac{dr}{ds}\right)ds$

$= \oint_C \left((\sin s\,\hat{i} + \cos s\,\hat{k})\cdot(-\sin s\,\hat{i} + \cos s\,\hat{j})\right)ds$

$= \int_0^{2\pi} -\sin^2 s\, ds = \int_0^{2\pi}\left(\frac{\cos 2s - 1}{2}\right)ds = \frac{1}{2}\left[\frac{\sin 2s}{2}\right]_0^{2\pi} - \frac{1}{2}[s]_0^{2\pi} = -\pi$ RHS

LHS $\iint_S (\mathrm{Curl} F)\cdot\hat{n}\,dA = \iint_R \left(\left(\frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z}\right)N_1 + \left(\frac{\partial F_1}{\partial z} - \frac{\partial F_3}{\partial x}\right)N_2 + \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right)N_3\right)du\,dv$

Now $\mathrm{Curl} F = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y & z & x \end{vmatrix} = -\hat{i} - \hat{j} - \hat{k}$

Now $N = \text{grad} f$, $f = x^2+y^2+z-1 \Rightarrow \text{grad} f = \frac{\partial f}{\partial x}\hat{i} + \frac{\partial f}{\partial y}\hat{j} + \frac{\partial f}{\partial z}\hat{k}$

$\Rightarrow \boxed{N = \text{grad} f = 2x\hat{i} + 2y\hat{j} + \hat{k}}$

Hence $\text{curl} F \cdot N = (-\hat{i}-\hat{j}-\hat{k})\cdot(2x\hat{i}+2y\hat{j}+\hat{k}) = -2x-2y-1$

Hence $\iint_S (\text{curl} F \cdot \hat{n}) dA = \iint_R (-2x-2y-1)\,dx\,dy$

$z = 1-(x^2+y^2)$
base is circle $x^2+y^2=1$

Using polar co-ordinates: $x = r\cos\theta$, $y = r\sin\theta$, $dx\,dy = r\,dr\,d\theta$

$\therefore R: r \le 1,\ 0 \le \theta \le 2\pi$

Hence $\iint_S \text{curl} F \cdot \hat{n}\, dA = \iint_R (-2x-2y-1)\,dx\,dy$

$$= \iint_R (-2r\cos\theta - 2r\sin\theta - 1)\, r\,dr\,d\theta$$

$$= \int_0^{2\pi}\left[\int_0^1 \left(-2r^2(\cos\theta+\sin\theta) - r\right)dr\right]d\theta$$

(θ - is const)

$$= \int_0^{2\pi}\left(-\frac{2}{3}\left[r^3\right]_0^1(\cos\theta+\sin\theta) - \left[\frac{r^2}{2}\right]_0^1\right)d\theta = \int_0^{2\pi}\left(-\frac{2}{3}(\sin\theta+\cos\theta) - \frac{1}{2}\right)d\theta$$

$$= -\frac{2}{3}\left\{\left[\cos\theta\right]_0^{2\pi} + \left[\sin\theta\right]_0^{2\pi}\right\} - \frac{1}{2}\left[\theta\right]_0^{2\pi} = -\frac{2}{3}(0) - \pi = -\pi \text{ LHS.}$$

$\therefore$ LHS = RHS $\Rightarrow$ Stokes them is proved. & value of integral $= -\pi$

<u>Example</u> Evaluate $\int_C F\cdot r'(s)\,ds$ using stoke's them, where $F = [-5y, 4x, z]$ & 'C' is the circle $x^2+y^2=4$, $z=1$

<u>Solⁿ</u> As per stoke's themᵐ $\int_C F\cdot r'(s)\,ds = \iint_S \text{curl} F \cdot \hat{n}\, dA$.

$F = -5y\hat{i} + 4x\hat{j} + z\hat{k} \Rightarrow \text{curl} F = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ -5y & 4x & z \end{vmatrix} = 0\cdot\hat{i} + 0\hat{j} + (4+5)\hat{k} = 9\hat{k}$.

Here the circle $x^2+y^2=4$ is on xy-plane $\Rightarrow \hat{n} = \hat{k}$.

$\text{curl} F\cdot\hat{n} = (9\hat{k})\cdot\hat{k} = 9$

Now $\oint_C F\cdot r'(s)\,ds = \iint_S (\text{curl} F\cdot\hat{n})\,dA = \iint_S 9\,dA = 9 \times \text{Area of } S = 9\times\pi\times 2^2$

$= 9\times 4\pi = 36\pi$

<u>N.B</u> Green's theorem is a special case of stokes themᵐ when $F = F_1\hat{i} + F_2\hat{j}$ & S is bounded by a simple closed curve C which is piecewise smooth in xy-plane. Then

$$\iint_S \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right)dx\,dy = \oint_C F_1\,dx + F_2\,dy$$

Assignments